A TEXT-BOOK

OF

EUCLID'S ELEMENTS

FOR THE USE OF SCHOOLS

BOOK I.

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PREFACE.

THIS edition of Euclid's First Book has been prepared in accordance with the wishes of many teachers. It consists merely of a reprint from our complete "Text book of Euclid's Elements" together with a small collection of Miscellaneous Examples. It will probably be found that these and the easy exercises interspersed throughout the text provide sufficient practice for beginners. Teachers who require more examples and problems will find a large number, carefully arranged and classified, on pages 87—119 of our complete edition.

H. S. HALL,
F. H. STEVENS.

May 1889.

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EUCLID'S ELEMENTS.

BOOK I.

Definitions.

1. A **point** is that which has position, but no magnitude.

2. A **line** is that which has length without breadth.

The extremities of a line are points, and the intersection of two lines is a point.

3. A **straight line** is that which lies evenly between its extreme points.

Any portion cut off from a straight line is called a **segment** of it.

4. A **surface** is that which has length and breadth, but no thickness.

The boundaries of a surface are lines.

5. A **plane surface** is one in which any two points being taken, the straight line between them lies wholly in that surface.

A plane surface is frequently referred to simply as a plane.

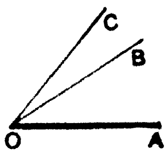
Note. Euclid regards a point merely as a *mark of position*, and he therefore attaches to it no idea of size and shape.

Similarly he considers that the properties of a line arise only from its *length* and *position*, without reference to that minute breadth which every line must really have *if actually drawn*, even though the most perfect instruments are used.

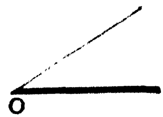
The definition of a surface is to be understood in a similar way.

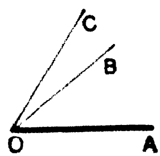
6. **A plane angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.**

The point at which the straight lines meet is called the **vertex** of the angle, and the straight lines themselves the **arms** of the angle.



When several angles are at one point O, any one of them is expressed by three letters, of which the letter that refers to the vertex is put between the other two. Thus if the straight lines OA, OB, OC meet at the point O, the angle contained by the straight lines OA, OB is named the angle AOB or BOA; and the angle contained by OA, OC is named the angle AOC or COA. Similarly the angle contained by OB, OC is referred to as the angle BOC or COB. But if there be only one angle at a point, it may be expressed by a single letter, as *the angle at* O.



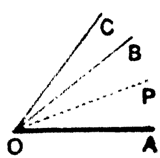


Of the two straight lines OB, OC shewn in the adjoining figure, we recognize that OC is *more inclined* than OB to the straight line OA: this we express by saying that the angle AOC is greater than the angle AOB. Thus an angle must be regarded as having *magnitude*.

It should be observed that the angle AOC is the sum of the angles AOB and BOC; and that AOB is the difference of the angles AOC and BOC.

The beginner is cautioned against supposing that the size of an angle is altered either by increasing or diminishing the length of its arms.

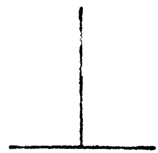
[Another view of an angle is recognized in many branches of mathematics; and though not employed by Euclid, it is here given because it furnishes more clearly than any other a conception of what is meant by the *magnitude* of an angle.



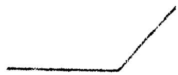
Suppose that the straight line OP in the figure is capable of revolution about the point O, like the hand of a watch, but in the opposite direction; and suppose that in this way it has passed successively from the position OA to the positions occupied by OB and OC.

Such a line must have undergone *more turning* in passing from OA to OC, than in passing from OA to OB; and consequently the angle AOC is said to be greater than the angle AOB.]

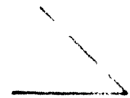
7. **When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.**



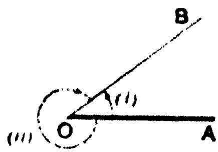
8. An **obtuse angle** is an angle which is greater than one right angle, but less than two right angles.



9. An **acute angle** is an angle which is less than a right angle.



[In the adjoining figure the straight line OB may be supposed to have arrived at its present position, from the position occupied by OA, by revolution about the point O in *either* of the two directions indicated by the arrows: thus two straight lines drawn from a point may be considered as forming *two* angles, (marked (i) and (ii) in the figure) of which the greater (ii) is said to be **reflex**.



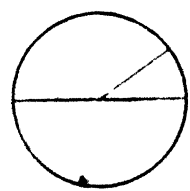
If the arms OA, OB are in the same straight line, the angle formed by them on either side is called a **straight angle**.]

10. Any portion of a plane surface bounded by one or more lines, straight or curved, is called a **plane figure**.

The sum of the bounding lines is called the **perimeter** of the figure.

Two figures are said to be equal in **area**, when they enclose equal portions of a plane surface.

11. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another: this point is called the **centre** of the circle.



A **radius** of a circle is a straight line drawn from the centre to the circumference.

12. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

13. A **semicircle** is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

14. A **segment of a circle** is the figure bounded by a straight line and the part of the circumference which it cuts off.

15. **Rectilineal figures** are those which are bounded by straight lines.

16. A **triangle** is a plane figure bounded by straight lines.

Any one of the angular points of a triangle may be regarded as its **vertex**; and the opposite side is then called the **base**.

17. A **quadrilateral** is a plane figure bounded by *four* straight lines.

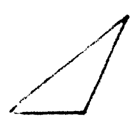
The straight line which joins opposite angular points in a quadrilateral is called a **diagonal**.

18. A **polygon** is a plane figure bounded by more than four straight lines.

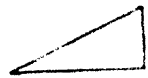
19. An **equilateral triangle** is a triangle whose three sides are equal.

20. An **isosceles triangle** is a triangle two of whose sides are equal.

21. A **scalene triangle** is a triangle which has three unequal sides.

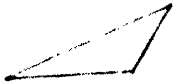


22. A **right-angled triangle** is a triangle which has a right angle.



The side opposite to the right angle in a right-angled triangle is called the **hypotenuse**.

23. An **obtuse-angled triangle** is a triangle which has an obtuse angle.

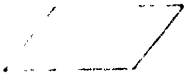


24. An **acute-angled triangle** is a triangle which has *three* acute angles.

[It will be seen hereafter (Book I. Proposition 17) *that every triangle must have at least two acute angles.*]

25. **Parallel straight lines** are such as, being in the same plane, do not meet, however far they are produced in either direction.

26. A **Parallelogram** is a four-sided figure which has its opposite sides parallel.



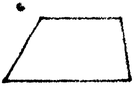
27. A **rectangle** is a parallelogram which has one of its angles a right angle.

28. A **square** is a four-sided figure which has all its sides equal and all its angles right angles.

[It may easily be shewn that if a quadrilateral has all its sides equal and *one* angle a right angle, then *all* its angles will be right angles.]

29. A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

30. A **trapezium** is a four-sided figure which has *two* of its sides parallel.



ON THE POSTULATES.

In order to effect the *constructions* necessary to the study of geometry, it must be supposed that certain instruments are available; but it has always been held that such instruments should be as few in number, and as simple in character as possible.

For the purposes of the first Six Books *a straight ruler* and a pair of compasses are all that are needed; and in the following **Postulates**, or requests, Euclid demands the use of such instruments, and assumes that they suffice, theoretically as well as practically, to carry out the processes mentioned below.

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point.

When we draw a straight line from the point A to the point B, we are said to *join* AB.

2. That a *finite*, that is to say, a terminated straight line may be produced to any length in that straight line.

3. That a circle may be described from any centre, at any distance from that centre, that is, with a radius equal to any finite straight line drawn from the centre.

It is important to notice that the Postulates include no means of *direct measurement*: hence the straight ruler is not supposed to be *graduated*; and the compasses, in accordance with Euclid's use, are not to be employed for *transferring distances* from one part of a figure to another.

ON THE AXIOMS.

The science of Geometry is based upon certain simple statements, the truth of which is assumed at the outset to be self-evident.

These self-evident truths, called by Euclid *Common Notions*, are now known as the **Axioms**.

The necessary characteristics of an Axiom are

(i) That it should be *self-evident;* that is, that its truth should be immediately accepted without proof.

(ii) That it should be *fundamental;* that is, that its truth should not be derivable from any other truth more simple than itself.

(iii) That it should supply a basis for the establishment of further truths.

These characteristics may be summed up in the following definition.

DEFINITION. An **Axiom** is a self-evident truth, which neither requires nor is capable of proof, but which serves as a foundation for future reasoning.

Axioms are of two kinds, *general* and *geometrical*.

General Axioms apply to *magnitudes of all kinds*. Geometrical Axioms refer exclusively to *geometrical magnitudes*, such as have been already indicated in the definitions.

GENERAL AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater of the unequals.

5. If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.

6. Things which are double of the same thing, or of equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

9.* The whole is greater than its part.

* To preserve the classification of general and geometrical axioms, we have placed Euclid's *ninth* axiom before the *eighth*.

GEOMETRICAL AXIOMS.

8. Magnitudes which can be made to coincide with one another, are equal.

This axiom affords the ultimate test of the equality of two geometrical magnitudes. It implies that any line, angle, or figure, may be supposed to be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison.

This process is called **superposition**, and the first magnitude is said to be **applied** to the other.

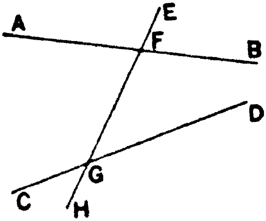
10. Two straight lines cannot enclose a space.

11. All right angles are equal.

[The statement that all right angles are equal, admits of proof, and is therefore perhaps out of place as an Axiom.]

12. If a straight line meet two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.

That is to say, if the two straight lines AB and CD are met by the straight line EH at F and G, in such a way that the angles BFG, DGF are together less than two right angles, it is asserted that AB and CD will meet if continually produced in the direction of B and D.



[Axiom 12 has been objected to on the double ground that it cannot be considered self-evident, and that its truth may be deduced from simpler principles. It is employed for the first time in the 29th Proposition of Book I., where a short discussion of the difficulty will be found.

The converse of this Axiom is proved in Book I. Proposition 17.]

INTRODUCTORY.

Plane Geometry deals with the properties of all lines and figures that may be drawn upon a plane surface.

Euclid in his first Six Books confines himself to the properties of straight lines, rectilineal figures, and circles.

The *Definitions* indicate the subject-matter of these books: the *Postulates and Axioms* lay down the fundamental principles which regulate all investigation and argument relating to this subject-matter.

Euclid's method of exposition divides the subject into a number of separate discussions, called **propositions**; each proposition, though in one sense complete in itself, is derived from results previously obtained, and itself leads up to subsequent propositions.

Propositions are of two kinds, **Problems** and **Theorems.**

A **Problem** proposes to effect some geometrical construction, such as to draw some particular line, or to construct some required figure.

A **Theorem** proposes to demonstrate some geometrical truth.

A Proposition consists of the following parts:

The General Enunciation, the Particular Enunciation, the Construction, and the Demonstration or Proof.

(i) The **General Enunciation** is a preliminary statement, describing in general terms the purpose of the proposition.

In a *problem* the Enunciation states the construction which it is proposed to effect: it therefore names first the **Data,** or things given, secondly the **Quæsita,** or things required.

In a *theorem* the Enunciation states the property which it is proposed to demonstrate: it names first, the **Hypothesis,** or the conditions assumed; secondly, the **Conclusion,** or the assertion to be proved.

(ii) The **Particular Enunciation** repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily.

(iii) The **Construction** then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem.

(iv) Lastly, the **Demonstration** proves that the object proposed in a problem has been accomplished, or that the property stated in a theorem is true.

Euclid's reasoning is said to be **Deductive**, because by a connected chain of argument it **deduces** new truths from truths already proved or admitted.

The initial letters Q. E. F., placed at the end of a problem, stand for **Quod erat Faciendum**, *which was to be done.*

The letters Q. E. D. are appended to a theorem, and stand for **Quod erat Demonstrandum**, *which was to be proved.*

A **Corollary** is a statement the truth of which follows readily from an established proposition; it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof.

The following symbols and abbreviations may be employed in writing out the propositions of Book I., though their use is not recommended to beginners.

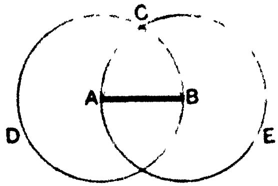
$\therefore$	*for*	therefore,	par^{l} (or $\parallel$)	*for*	parallel,
$=$	,,	is, or are, equal to,	par^{m}	,,	parallelogram,
$\angle$	,,	angle,	sq.	,,	square,
rt. $\llcorner$	,,	right angle,	rectil.	,,	rectilineal,
$\triangle$	,,	triangle,	st. line	,,	straight line,
perp.	,,	perpendicular,	pt.	,,	point;

and all obvious contractions of words, such as opp., adj., diag., &c., for opposite, adjacent, diagonal, &c.

SECTION I.

Proposition 1. Problem.

To describe an equilateral triangle on a given finite straight line.



Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

Construction. From centre A, with radius AB, describe the circle BCD. *Post.* 3.

From centre B, with radius BA, describe the circle ACE. *Post.* 3.

From the point C at which the circles cut one another, draw the straight lines CA and CB to the points A and B. *Post.* 1.

Then shall ABC be an equilateral triangle.

Proof. Because A is the centre of the circle BCD,
therefore AC is equal to AB. *Def.* 11.

And because B is the centre of the circle ACE,
therefore BC is equal to BA. *Def.* 11.

But it has been shewn that AC is equal to AB;
therefore AC and BC are each equal to AB.

But things which are equal to the same thing are equal to one another. *Ax.* 1.

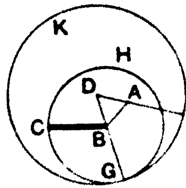
Therefore AC is equal to BC.

Therefore CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral;
and it is described on the given straight line AB. Q.E.F.

PROPOSITION 2. PROBLEM.

From a given point to draw a straight line equal to a given straight line.



Let A be the given point, and BC the given straight line.

It is required to draw from the point A a straight line equal to BC.

Construction. Join AB; *Post.* 1.

and on AB describe an equilateral triangle DAB. I. 1.

From centre B, with radius BC, describe the circle CGH. *Post.* 3.

Produce DB to meet the circle CGH at G. *Post.* 2.

From centre D, with radius DG, describe the circle GKF.

Produce DA to meet the circle GKF at F. *Post.* 2.

Then AF shall be equal to BC.

Proof. Because B is the centre of the circle CGH,

therefore BC is equal to BG. *Def.* 11.

And because D is the centre of the circle GKF,

therefore DF is equal to DG; *Def.* 11.

and DA, DB, parts of them are equal; *Def.* 19.

therefore the remainder AF is equal to the remainder BG. *Ax.* 3.

And it has been shewn that BC is equal to BG;

therefore AF and BC are each equal to BG.

But things which are equal to the same thing are equal to one another. *Ax.* 1.

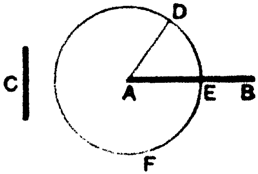
Therefore AF is equal to BC;

and it has been drawn from the given point A. Q. E. F.

[This Proposition is rendered necessary by the restriction, tacitly imposed by Euclid, that compasses shall not be used to *transfer distances*.]

PROPOSITION 3. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.



Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB a part equal to C.

Construction. From the point A draw the straight line AD equal to C; I. 2.
and from centre A, with radius AD, describe the circle DEF, meeting AB at E. *Post.* 3.

Then AE shall be equal to C.

Proof. Because A is the centre of the circle DEF,
therefore AE is equal to AD. *Def.* 11.
But C is equal to AD. *Constr.*
Therefore AE and C are each equal to AD.
Therefore AE is equal to C;
and it has been cut off from the given straight line AB.

Q.E.F.

EXERCISES.

1. On a given straight line describe an isosceles triangle having each of the equal sides equal to a given straight line.

2. On a given base describe an isosceles triangle having each of the equal sides double of the base.

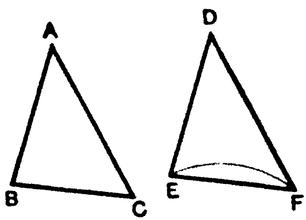
3. In the figure of I. 2, if AB is equal to BC, shew that D, the vertex of the equilateral triangle, will fall on the circumference of the circle CGH.

Obs. Every triangle has six **parts**, namely its three sides and three angles.

Two triangles are said to be **equal in all respects**, when they can be made to coincide with one another by *superposition* (see note on Axiom 8), and in this case each part of the one is equal to a corresponding part of the other.

PROPOSITION 4. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal; then shall their bases or third sides be equal, and the triangles shall be equal in area, and their remaining angles shall be equal, each to each, namely those to which the equal sides are opposite: that is to say, the triangles shall be equal in all respects.



Let ABC, DEF be two triangles, which have the side AB equal to the side DE, the side AC equal to the side DF, and the contained angle BAC equal to the contained angle EDF.

Then shall the base BC be equal to the base EF, and the triangle ABC shall be equal to the triangle DEF in area; and the remaining angles shall be equal, each to each, to which the equal sides are opposite,

namely the angle ABC to the angle DEF,
and the angle ACB to the angle DFE.

For if the triangle ABC be applied to the triangle DEF,
so that the point A may be on the point D,
and the straight line AB along the straight line DE,
then because AB is equal to DE, *Hyp.*
therefore the point B must coincide with the point E.

And because AB falls along DE,
and the angle BAC is equal to the angle EDF, *Hyp.*
therefore AC must fall along DF.
And because AC is equal to DF, *Hyp.*
therefore the point C must coincide with the point F.
Then B coinciding with E, and C with F,
the base BC must coincide with the base EF;
for if not, two straight lines would enclose a space; which is impossible. *Ax.* 10.

Thus the base BC coincides with the base EF, and is therefore equal to it. *Ax.* 8.

And the triangle ABC coincides with the triangle DEF, and is therefore equal to it in area. *Ax.* 8.

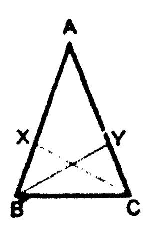
And the remaining angles of the one coincide with the remaining angles of the other, and are therefore equal to them,
namely, the angle ABC to the angle DEF,
and the angle ACB to the angle DFE.

That is, the triangles are equal in all respects. Q. E. D.

NOTE. It follows that two triangles which are equal in their several parts are equal also in *area;* but it should be observed that equality of area in two triangles does not necessarily imply equality in their several parts: that is to say, triangles may be equal in *area*, without being of the same *shape*.

Two triangles which are equal in all respects have *identity of form and magnitude*, and are therefore said to be **identically equal**, or **congruent**.

The following application of Proposition 4 anticipates the chief difficulty of Proposition 5.



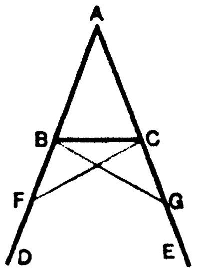
In the equal sides AB, AC of an isosceles triangle ABC, the points X and Y are taken, so that AX is equal to AY; and BY and CX are joined.

Shew that BY is equal to CX.

In the two triangles XAC, YAB,
XA is equal to YA, and AC is equal to AB; *Hyp.*
that is, the two sides XA, AC are equal to the two sides YA, AB, each to each;
and the angle at A, which is contained by these sides, is common to both triangles:
therefore the triangles are equal in all respects; I. 4.
so that XC is equal to YB. Q. E. D.

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles on the other side of the base shall also be equal to one another.



Let ABC be an isosceles triangle, having the side AB equal to the side AC, and let the straight lines AB, AC be produced to D and E:

then shall the angle ABC be equal to the angle ACB, and the angle CBD to the angle BCE.

Construction. In BD take any point F;
and from AE the greater cut off AG equal to AF the less. I. 3.
Join FC, GB.

Proof. Then in the triangles FAC, GAB,

Because { FA is equal to GA, *Constr.*
and AC is equal to AB, *Hyp.*
also the contained angle at A is common to the two triangles;

therefore the triangle FAC is equal to the triangle GAB in all respects; I. 4.
that is, the base FC is equal to the base GB,
and the angle ACF is equal to the angle ABG,
also the angle AFC is equal to the angle AGB.

Again, because the whole AF is equal to the whole AG,
of which the parts AB, AC are equal, *Hyp.*
therefore the remainder BF is equal to the remainder CG.

Then in the two triangles BFC, CGB,

Because { BF is equal to CG, *Proved.*
and FC is equal to GB, *Proved.*
also the contained angle BFC is equal to the contained angle CGB, *Proved.* }

therefore the triangles BFC, CGB are equal in all respects;
so that the angle FBC is equal to the angle GCB,
and the angle BCF to the angle CBG. I. 4.

Now it has been shewn that the whole angle ABG is equal to the whole angle ACF,
and that parts of these, namely the angles CBG, BCF, are also equal;
therefore the remaining angle ABC is equal to the remaining angle ACB;
and these are the angles at the base of the triangle ABC.
Also it has been shewn that the angle FBC is equal to the angle GCB;
and these are the angles on the other side of the base. Q.E.D.

COROLLARY. *Hence if a triangle is equilateral it is also equiangular.*

EXERCISES.

1. AB is a given straight line and C a given point outside it: shew how to find any points in AB such that their distance from C shall be equal to a given length L. Can such points always be found?

2. If the vertex C and one extremity A of the base of an isosceles triangle are given, find the other extremity B, supposing it to lie on a given straight line PQ.

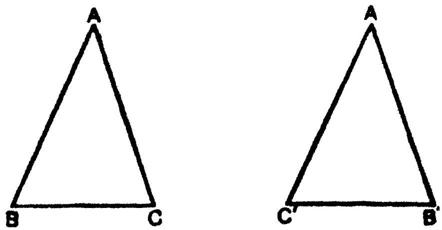
3. Describe a rhombus having given two opposite angular points A and C, and the length of each side.

4. AMNB is a straight line; on AB describe a triangle ABC such that the side AC shall be equal to AN and the side BC to MB.

5. In Prop. 2 the point A may be joined to *either* extremity of BC. Draw the figure and prove the proposition in the case when A is joined to C.

The following proof is sometimes given as a substitute for the first part of Proposition 5:

PROPOSITION 5. ALTERNATIVE PROOF.



Let ABC be an isosceles triangle, having AB equal to AC:
then shall the angle ABC be equal to the angle ACB.

Suppose the triangle ABC to be taken up, turned over and laid down again in the position A'B'C', where A'B', A'C', B'C' represent the new positions of AB, AC, BC.

Then A'B' is equal to A'C'; and A'B' is AB in its new position,
therefore AB is equal to A'C';
in the same way AC is equal to A'B';
and the included angle BAC is equal to the included angle C'A'B', for they are the same angle in different positions;
therefore the triangle ABC is equal to the triangle A'C'B' in all respects:
so that the angle ABC is equal to the angle A'C'B'. I. 4.
But the angle A'C'B' is the angle ACB in its new position;
therefore the angle ABC is equal to the angle ACB.

Q. E. D.

EXERCISES.

CHIEFLY ON PROPOSITIONS 4 AND 5.

1. Two circles have the same centre O; OAD and OBE are straight lines drawn to cut the smaller circle in A and B and the larger circle in D and E: prove that

(i) AD = BE. (ii) DB = EA.

(iii) The angle DAB is equal to the angle EBA.

(iv) The angle ODB is equal to the angle OEA.

2. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.

(iii) AN = AM. (iv) BN = DM.

[Draw a separate figure in each case.]

3. O is the centre of a circle and OA, OB are radii ; OM divides the angle AOB into two equal parts and cuts the line AB in M : prove that AM = BM.

4. ABC, DBC are two isosceles triangles described on the same base BC but on opposite sides of it : prove that the angle ABD is equal to the angle ACD.

5. ABC, DBC are two isosceles triangles described on the same base BC, but on opposite sides of it : prove that if AD be joined, each of the angles BAC, BDC will be divided into two equal parts.

6. PQR, SQR are two isosceles triangles described on the same base QR, and on the same side of it : shew that the angle PQS is equal to the angle PRS, and that the line PS divides the angle QPR into two equal parts.

7. If in the figure of Exercise 5 the line AD meets BC in E, prove that BE = EC.

8. ABCD is a rhombus and AC is joined : prove that the angle DAB is equal to the angle DCB.

9. ABCD is a quadrilateral having the opposite sides BC, AD equal, and also the angle BCD equal to the angle ADC : prove that BD is equal to AC.

10. AB, AC are the equal sides of an isosceles triangle ; L, M, N are the middle points of AB, BC, and CA respectively : prove that LM = MN.

Prove also that the angle ALM is equal to the angle ANM.

Definition. Each of two Theorems is said to be the **Converse** of the other, when the hypothesis of each is the conclusion of the other.

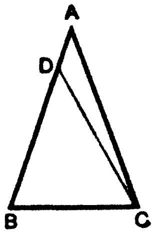
It will be seen, on comparing the hypotheses and conclusions of Props. 5 and 6, that each proposition is the converse of the other.

Note. Proposition 6 furnishes the first instance of an *indirect method of proof*, frequently used by Euclid. It consists in shewing that an absurdity must result from supposing the theorem to be otherwise than true. This form of demonstration is known as the **Reductio ad Absurdum**, and is most commonly employed in establishing the converse of some foregoing theorem.

It must not be supposed that the converse of a true theorem is itself necessarily true : for instance, it will be seen from Prop. 8, Cor. that if two triangles have their sides equal, each to each, then their angles will also be equal, each to each ; but it may easily be shewn by means of a figure that the converse of this theorem is not necessarily true.

PROPOSITION 6. THEOREM.

If two angles of a triangle be equal to one another, then the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.



Let ABC be a triangle, having the angle ABC equal to the angle ACB :

then shall the side AC be equal to the side AB.

Construction. For if AC be not equal to AB,
one of them must be greater than the other.
If possible, let AB be the greater ;
and from it cut off BD equal to AC. I. 3.
Join DC.

Proof. Then in the triangles DBC, ACB,

Because { DB is equal to AC, *Constr.*
and BC is common to both,
also the contained angle DBC is equal to the contained angle ACB ; *Hyp.* }

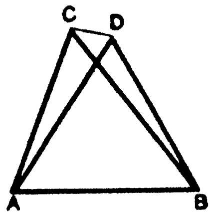
therefore the triangle DBC is equal in area to the triangle ACB, I. 4.
the part equal to the whole ; which is absurd. *Ax.* 9.

Therefore AB is not unequal to AC ;
that is, AB is equal to AC. Q.E.D.

COROLLARY. *Hence if a triangle is equiangular it is also equilateral.*

PROPOSITION 7. THEOREM.

On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.



If it be possible, on the same base AB, and on the same side of it, let there be two triangles ACB, ADB, having their sides AC, AD, which are terminated at A, equal to one another, and likewise their sides BC, BD, which are terminated at B, equal to one another.

CASE I. When the vertex of each triangle is without the other triangle.

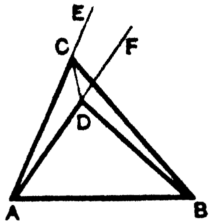
Construction. Join CD. *Post.* 1.

Proof. Then in the triangle ACD,
because AC is equal to AD, *Hyp.*
therefore the angle ACD is equal to the angle ADC. I. 5.

But the whole angle ACD is greater than its part, the angle BCD,
therefore also the angle ADC is greater than the angle BCD;
still more then is the angle BDC greater than the angle BCD.

Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5
but it was shewn to be greater; which is impossible.

CASE II. When one of the vertices, as D, is within the other triangle ACB.



Construction. As before, join CD; *Post.* 1.
and produce AC, AD to E and F. *Post.* 2.
Then in the triangle ACD, because AC is equal to AD, *Hyp.*
therefore the angles ECD, FDC, on the other side of the base, are equal to one another. I. 5.
But the angle ECD is greater than its part, the angle BCD;
therefore the angle FDC is also greater than the angle BCD:
still more then is the angle BDC greater than the angle BCD.
Again, in the triangle BCD,
because BC is equal to BD, *Hyp.*
therefore the angle BDC is equal to the angle BCD: I. 5.
but it has been shewn to be greater; which is impossible.

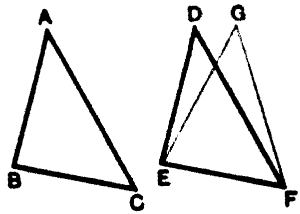
The case in which the vertex of one triangle is on a side of the other needs no demonstration.

Therefore AC cannot be equal to AD, and *at the same time*, BC equal to BD. Q.E.D.

NOTE. The sides AC, AD are called **conterminous** sides; similarly the sides BC, BD are conterminous.

PROPOSITION 8. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, then the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides of the other.



Let ABC, DEF be two triangles, having the two sides BA, AC equal to the two sides ED, DF, each to each, namely BA to ED, and AC to DF, and also the base BC equal to the base EF:

then shall the angle BAC be equal to the angle EDF.

Proof. For if the triangle ABC be applied to the triangle DEF, so that the point B may be on E, and the straight line BC along EF;

then because BC is equal to EF, *Hyp.*

therefore the point C must coincide with the point F.

Then, BC coinciding with EF,

it follows that BA and AC must coincide with ED and DF: for if not, they would have a different situation, as EG, GF: then, on the same base and on the same side of it there would be two triangles having their *conterminous* sides equal.

But this is impossible. I. 7.

Therefore the sides BA, AC coincide with the sides ED, DF.

That is, the angle BAC coincides with the angle EDF, and is therefore equal to it. *Ax.* 8.

Q. E. D.

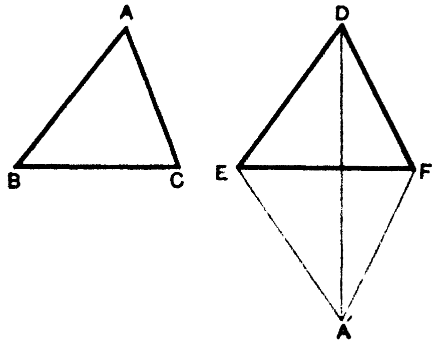
NOTE. In this Proposition the three sides of one triangle are given equal respectively to the three sides of the other; and from this it is shewn that the two triangles may be made *to coincide with one another.*

Hence we are led to the following important Corollary.

COROLLARY. *If in two triangles the three sides of the one are equal to the three sides of the other, each to each, then the triangles are equal in all respects.*

The following proof of Prop. 8 is worthy of attention as it is independent of Prop. 7, which frequently presents difficulty to a beginner.

PROPOSITION 8. ALTERNATIVE PROOF.



Let ABC and DEF be two triangles, which have the sides BA, AC equal respectively to the sides ED, DF, and the base BC equal to the base EF:

then shall the angle BAC be equal to the angle EDF.

For apply the triangle ABC to the triangle DEF, so that B may fall on E, and BC along EF, and so that the point A may be on the side of EF remote from D,

then C must fall on F, since BC is equal to EF.

Let A'EF be the new position of the triangle ABC.

If neither DF, FA' nor DE, EA' are in one straight line,

join DA'.

CASE I. When DA' intersects EF.

Then because ED is equal to EA',

therefore the angle EDA' is equal to the angle EA'D. I. 5.

Again because FD is equal to FA',

therefore the angle FDA' is equal to the angle FA'D. I. 5.

Hence the whole angle EDF is equal to the whole angle EA'F;

that is, the angle EDF is equal to the angle BAC.

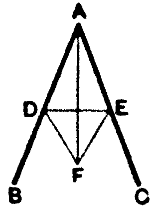
Two cases remain which may be dealt with in a similar manner: namely,

CASE II. When DA' meets EF produced.

CASE III. When one pair of sides, as DF, FA', are in one straight line.

PROPOSITION 9. PROBLEM.

To bisect a given angle, that is, to divide it into two equal parts.



Let BAC be the given angle:
it is required to bisect it.

Construction. In AB take any point D;
and from AC cut off AE equal to AD. I. 3.
Join DE;
and on DE, on the side remote from A, describe an equilateral triangle DEF. I. 1.
Join AF.
Then shall the straight line AF bisect the angle BAC.

Proof. For in the two triangles DAF, EAF,

Because
- DA is equal to EA, *Constr.*
- and AF is common to both;
- and the third side DF is equal to the third side EF; *Def.* 19.

therefore the angle DAF is equal to the angle EAF. I. 8.
Therefore the given angle BAC is bisected by the straight line AF. Q.E.F.

EXERCISES.

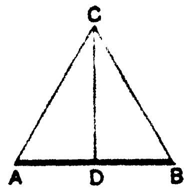
1. If in the above figure the equilateral triangle DFE were described on the same side of DE as A, what different cases would arise? And under what circumstances would the construction fail?

2. In the same figure, shew that AF also bisects the angle DFE.

3. Divide an angle into four equal parts.

PROPOSITION 10. PROBLEM.

To bisect a given finite straight line, that is, to divide it into two equal parts.



Let AB be the given straight line:
it is required to divide it into two equal parts.

Constr. On AB describe an equilateral triangle ABC, I. 1.
and bisect the angle ACB by the straight line CD, meeting AB at D. I. 9.

Then shall AB be bisected at the point D.

Proof. For in the triangles ACD, BCD,

Because { AC is equal to BC, *Def.* 19.
and CD is common to both;
also the contained angle ACD is equal to the contained angle BCD; *Constr.* }

Therefore the triangles are equal in all respects:
so that the base AD is equal to the base BD. I. 4.
Therefore the straight line AB is bisected at the point D.

Q. E. F.

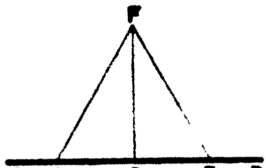
EXERCISES.

1. Shew that the straight line which bisects the vertical angle of an isosceles triangle, also bisects the base.

2. On a given base describe an isosceles triangle such that the sum of its equal sides may be equal to a given straight line.

PROPOSITION 11. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.



Let AB be the given straight line, and C the given point in it.

It is required to draw from the point C **a straight line at right angles to AB.**

Construction. In AC take any point D,
and from CB cut off CE equal to CD. I. 3.
On DE describe the equilateral triangle DFE. I. 1.
Join CF.

Then shall the straight line CF be at right angles to AB.

Proof. For in the triangles DCF, ECF,

Because { DC is equal to EC, *Constr.*
and CF is common to both;
and the third side DF is equal to the third side EF: *Def.* 19. }

Therefore the angle DCF is equal to the angle ECF: I. 8.
and these are adjacent angles.

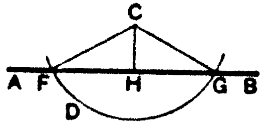
But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle; *Def.* 7.
therefore each of the angles DCF, ECF is a right angle.
Therefore CF is at right angles to AB,
and has been drawn from a point C in it. Q.E.F.

EXERCISE.

In the figure of the above proposition, shew that *any* point in FC, or FC produced, is equidistant from D and E.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.



Let AB be the given straight line, which may be produced in either direction, and let C be the given point without it.

It is required to draw from the point C a straight line perpendicular to AB.

Construction. On the side of AB remote from C take any point D;

and from centre C, with radius CD, describe the circle FDG, meeting AB at F and G. *Post.* 3.

Bisect FG at H; I. 10.

and join CH.

Then shall the straight line CH be perpendicular to AB.

Join CF and CG.

Proof. Then in the triangles FHC, GHC,

Because { FH is equal to GH, *Constr.*
and HC is common to both;
and the third side CF is equal to the third side CG, being radii of the circle FDG; *Def.* 11.

therefore the angle CHF is equal to the angle CHG; I. 8.

and these are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of these angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it.

Therefore CH is a perpendicular drawn to the given straight line AB from the given point C without it. Q. E. F.

NOTE. The given straight line AB must be of unlimited length, that is, it must be capable of production to an indefinite length in either direction, to ensure its being intersected in two points by the circle FDG.

EXERCISES ON PROPOSITIONS 1 TO 12.

1. Shew that the straight line which joins the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base.

2. Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another.

3. Two given points in the base of an isosceles triangle are equidistant from the extremities of the base: shew that they are also equidistant from the vertex.

4. If the opposite sides of a quadrilateral are equal, shew that the opposite angles are also equal.

5. Any two isosceles triangles XAB, YAB stand on the same base AB: shew that the angle XAY is equal to the angle XBY; and that the angle AXY is equal to the angle BXY.

6. Shew that the opposite angles of a rhombus are bisected by the diagonal which joins them.

7. Shew that the straight lines which bisect the base angles of an isosceles triangle form with the base a triangle which is also isosceles.

8. ABC is an isosceles triangle having AB equal to AC; and the angles at B and C are bisected by straight lines which meet at O: shew that OA bisects the angle BAC.

9. Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

10. The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF; and FB, EC are joined: shew that FB is equal to EC.

11. Shew that the diagonals of a rhombus bisect one another at right angles.

12. In the equal sides AB, AC of an isosceles triangle ABC two points X and Y are taken, so that AX is equal to AY; and CX and BY are drawn intersecting in O: shew that

(i) the triangle BOC is isosceles;
(ii) AO bisects the vertical angle BAC;
(iii) AO, if produced, bisects BC at right angles.

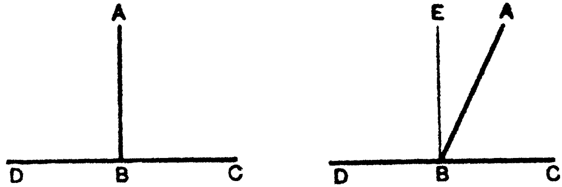
13. Describe an isosceles triangle, having given the base and the length of the perpendicular drawn from the vertex to the base.

14. In a given straight line find a point that is equidistant from two given points.

In what case is this impossible?

PROPOSITION 13. THEOREM.

If one straight line stand upon another straight line, then the adjacent angles shall be either two right angles, or together equal to two right angles.



Let the straight line AB stand upon the straight line DC: then the adjacent angles DBA, ABC shall be either two right angles, or together equal to two right angles.

CASE I. For if the angle DBA is equal to the angle ABC, each of them is a right angle. *Def.* 7.

CASE II. But if the angle DBA is not equal to the angle ABC,

from B draw BE at right angles to CD. I. 11.

Proof. Now the angle DBA is made up of the two angles DBE, EBA;

to each of these equals add the angle ABC;

then the two angles DBA, ABC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

Again, the angle EBC is made up of the two angles EBA, ABC;

to each of these equals add the angle DBE.

Then the two angles DBE, EBC are together equal to the three angles DBE, EBA, ABC. *Ax.* 2.

But the two angles DBA, ABC have been shewn to be equal to the same three angles;

therefore the angles DBA, ABC are together equal to the angles DBE, EBC. *Ax.* 1.

But the angles DBE, EBC are two right angles; *Constr.*

therefore the angles DBA, ABC are together equal to two right angles. Q. E. D.

DEFINITIONS.

(i) The **complement** of an acute angle is its *defect from* a right angle, that is, the angle by which it falls short of a right angle.

Thus two angles are **complementary**, when their sum is a right angle.

(ii) The **supplement** of an angle is its *defect from* two right angles, that is, the angle by which it falls short of two right angles.

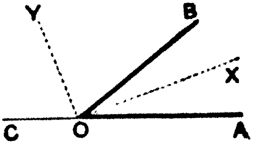
Thus two angles are **supplementary**, when their sum is two right angles.

COROLLARY. *Angles which are complementary or supplementary to the same angle are equal to one another.*

EXERCISES.

1. If the two exterior angles formed by producing a side of a triangle both ways are equal, shew that the triangle is isosceles.

2. *The bisectors of the adjacent angles which one straight line makes with another contain a right angle.*



NOTE. In the adjoining figure AOB is a given angle; and one of its arms AO is produced to C: the adjacent angles AOB, BOC are bisected by OX, OY.

Then OX and OY are called respectively the **internal** and **external bisectors** of the angle AOB.

Hence Exercise 2 may be thus enunciated:

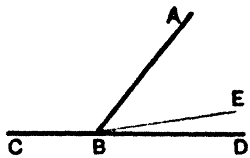
The internal and external bisectors of an angle are at right angles to one another.

3. **Shew that the angles AOX and COY are complementary.**

4. **Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.**

PROPOSITION 14. THEOREM.

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines shall be in one and the same straight line.



At the point B in the straight line AB, let the two straight lines BC, BD, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles:

then BD shall be in the same straight line with BC.

Proof. For if BD be not in the same straight line with BC, if possible, let BE be in the same straight line with BC.

Then because AB meets the straight line CBE,

therefore the adjacent angles CBA, ABE are together equal to two right angles. I. 13.

But the angles CBA, ABD are also together equal to two right angles. *Hyp.*

Therefore the angles CBA, ABE are together equal to the angles CBA, ABD. *Ax.* 11.

From each of these equals take the common angle CBA;

then the remaining angle ABE is equal to the remaining angle ABD; the part equal to the whole; which is impossible.

Therefore BE is not in the same straight line with BC.

And in the same way it may be shewn that no other line but BD can be in the same straight line with BC.

Therefore BD is in the same straight line with BC. Q.E.D.

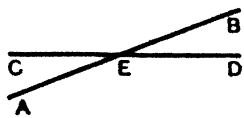
EXERCISE.

ABCD is a rhombus; and the diagonal AC is bisected at O. If O is joined to the angular points B and D; shew that OB and OD are in one straight line.

Obs. When two straight lines intersect at a point, four angles are formed; and any two of these angles *which are not adjacent*, are said to be **vertically opposite** to one another.

PROPOSITION 15. THEOREM.

If two straight lines intersect one another, then the vertically opposite angles shall be equal.



Let the two straight lines AB, CD cut one another at the point E:

then shall the angle AEC be equal to the angle DEB,

and the angle CEB to the angle AED

Proof. Because AE makes with CD the adjacent angles CEA, AED,

therefore these angles are together equal to two right angles. I. 13.

Again, because DE makes with AB the adjacent angles AED, DEB,

therefore these also are together equal to two right angles.

Therefore the angles CEA, AED are together equal to the angles AED, DEB.

From each of these equals take the common angle AED;

then the remaining angle CEA is equal to the remaining angle DEB. *Ax.* 3.

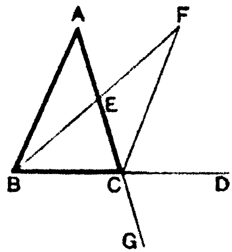
In a similar way it may be shewn that the angle CEB is equal to the angle AED. Q. E. D.

COROLLARY 1. *From this it is manifest that, if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.*

COROLLARY 2. *Consequently, when any number of straight lines meet at a point, the sum of the angles made by consecutive lines is equal to four right angles.*

PROPOSITION 16. THEOREM.

If one side of a triangle be produced, then the exterior angle shall be greater than either of the interior opposite angles.



Let ABC be a triangle, and let one side BC be produced to D: then shall the exterior angle ACD be greater than either of the interior opposite angles CBA, BAC.

Construction. Bisect AC at E: I. 10.
Join BE; and produce it to F, making EF equal to BE. I. 3.
Join FC.

Proof. Then in the triangles AEB, CEF,

Because { AE is equal to CE, *Constr.*
and EB to EF; *Constr.*
also the angle AEB is equal to the vertically opposite angle CEF; I. 15.

therefore the triangle AEB is equal to the triangle CEF in all respects: I. 4.

so that the angle BAE is equal to the angle ECF.

But the angle ECD is greater than its part, the angle ECF;
therefore the angle ECD is greater than the angle BAE;
that is, the angle ACD is greater than the angle BAC.

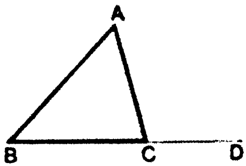
In a similar way, if BC be bisected, and the side AC produced to G, it may be shewn that the angle BCG is greater than the angle ABC.

But the angle BCG is equal to the angle ACD: I. 15.
therefore also the angle ACD is greater than the angle ABC.

Q. E. D.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.



Let ABC be a triangle: then shall any two of its angles, as ABC, ACB, be together less than two right angles.

Construction. Produce the side BC to D.

Proof. Then because ACD is an exterior angle of the triangle ABC,

therefore it is greater than the interior opposite angle ABC. I. 16.

To each of these add the angle ACB:

then the angles ACD, ACB are together greater than the angles ABC, ACB. *Ax.* 4.

But the adjacent angles ACD, ACB are together equal to two right angles. I. 13.

Therefore the angles ABC, ACB are together less than two right angles.

Similarly it may be shewn that the angles BAC, ACB, as also the angles CAB, ABC, are together less than two right angles. Q. E. D.

NOTE. It follows from this Proposition that *every triangle must have at least two acute angles:* for if one angle is obtuse, or a right angle, each of the other angles must be less than a right angle.

EXERCISES.

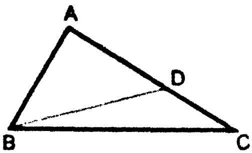
1. Enunciate this Proposition so as to shew that it is the converse of Axiom 12.

2. If any side of a triangle is produced both ways, the exterior angles so formed are together greater than two right angles.

3. Shew how a proof of Proposition 17 may be obtained by joining each vertex in turn to any point in the opposite side.

Proposition 18. Theorem.

If one side of a triangle be greater than another, then the angle opposite to the greater side shall be greater than the angle opposite to the less.



Let ABC be a triangle, in which the side AC is greater than the side AB :

then shall the angle ABC be greater than the angle ACB.

Construction. From AC, the greater, cut off a part AD equal to AB. I. 3.

Join BD.

Proof. Then in the triangle ABD,

because AB is equal to AD,

therefore the angle ABD is equal to the angle ADB. I. 5

But the exterior angle ADB of the triangle BDC is greater than the interior opposite angle DCB, that is, greater than the angle ACB. I. 16.

Therefore also the angle ABD is greater than the angle ACB; still more then is the angle ABC greater than the angle ACB. Q. E. D.

Euclid enunciated Proposition 18 as follows:

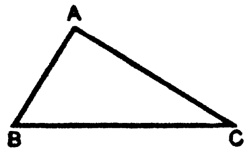
The greater side of every triangle has the greater angle opposite to it.

[This form of enunciation is found to be a common source of difficulty with beginners, who fail to distinguish what is *assumed* in it and what is *to be proved*.]

[For Exercises see page 38.]

PROPOSITION 19. THEOREM.

If one angle of a triangle be greater than another, then the side opposite to the greater angle shall be greater than the side opposite to the less.



Let ABC be a triangle in which the angle ABC is greater than the angle ACB :

then shall the side AC be greater than the side AB.

Proof. For if AC be not greater than AB,

it must be either equal to, or less than AB.

But AC is not equal to AB,

for then the angle ABC would be equal to the angle ACB ; I. 5.

but it is not. *Hyp.*

Neither is AC less than AB ;

for then the angle ABC would be less than the angle ACB ; I. 18.

but it is not : *Hyp.*

Therefore AC is neither equal to, nor less than AB.

That is, AC is greater than AB. Q.E.D.

NOTE. The mode of demonstration used in this Proposition is known as the **Proof by Exhaustion**. It is applicable to cases in which one of certain mutually exclusive suppositions must necessarily be true; and it consists in shewing the falsity of each of these suppositions in turn *with one exception*: hence the truth of the remaining supposition is inferred.

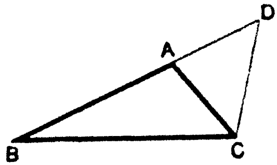
Euclid enunciated Proposition 19 as follows :

The greater angle of every triangle is subtended by the greater side, or, has the greater side opposite to it.

[For **Exercises** see page 38.]

Proposition 20. Theorem.

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle:
then shall any two of its sides be together greater than the third side:

namely, BA, AC, shall be greater than CB;
AC, CB greater than BA;
and CB, BA greater than AC.

Construction. Produce BA to the point D, making AD equal to AC. I. 3.

Join DC.

Proof. Then in the triangle ADC,
because AD is equal to AC, *Constr.*
therefore the angle ACD is equal to the angle ADC. I. 5.
But the angle BCD is greater than the angle ACD; *Ax.* 9.
therefore also the angle BCD is greater than the angle ADC,
that is, than the angle BDC.

And in the triangle BCD,
because the angle BCD is greater than the angle BDC, *Pr.*
therefore the side BD is greater than the side CB. I. 19.

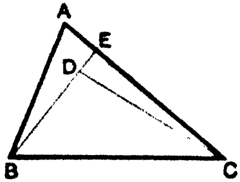
But BA and AC are together equal to BD;
therefore BA and AC are together greater than CB.

Similarly it may be shewn
that AC, CB are together greater than BA;
and CB, BA are together greater than AC. Q. E. D.

[For Exercises see page 88.]

PROPOSITION 21. THEOREM.

If from the ends of a side of a triangle, there be drawn two straight lines to a point within the triangle, then these straight lines shall be less than the other two sides of the triangle, but shall contain a greater angle.



Let ABC be a triangle, and from B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to a point D within the triangle:
then (i) BD and DC shall be together less than BA and AC;
(ii) the angle BDC shall be greater than the angle BAC.

Construction. Produce BD to meet AC in E.

Proof. (i) In the triangle BAE, the two sides BA, AE are together greater than the third side BE: I. 20.
to each of these add EC;
then BA, AC are together greater than BE, EC. *Ax.* 4.

Again, in the triangle DEC, the two sides DE, EC are together greater than DC: I. 20.
to each of these add BD;
then BE, EC are together greater than BD, DC.

But it has been shewn that BA, AC are together greater than BE, EC:
still more then are BA, AC greater than BD, DC.

(ii) Again, the exterior angle BDC of the triangle DEC is greater than the interior opposite angle DEC; I. 16.
and the exterior angle DEC of the triangle BAE is greater than the interior opposite angle BAE, that is, than the angle BAC; I. 16.
still more then is the angle BDC greater than the angle BAC.

Q.E.D.

EXERCISES

ON PROPOSITIONS 18 AND 19.

1. The hypotenuse is the greatest side of a right-angled triangle.

2. If two angles of a triangle are equal to one another, the sides also, which subtend the equal angles, are equal to one another. Prop. 6. Prove this indirectly by using the result of Prop. 18.

3. BC, the base of an isosceles triangle ABC, is produced to any point D; shew that AD is greater than either of the equal sides.

4. If in a quadrilateral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle.

5. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB.

6. ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively: shew that, if AB is greater than AC, then OB is greater than OC.

ON PROPOSITION 20.

7. The difference of any two sides of a triangle is less than the third side.

8. In a quadrilateral, if two opposite sides which are not parallel are produced to meet one another; shew that the perimeter of the greater of the two triangles so formed is greater than the perimeter of the quadrilateral.

9. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

10. The perimeter of a quadrilateral is greater than the sum of its diagonals.

11. Obtain a proof of Proposition 20 by bisecting an angle by a straight line which meets the opposite side.

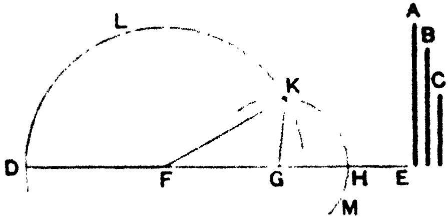
ON PROPOSITION 21.

12. In Proposition 21 shew that the angle BDC is greater than the angle BAC by joining AD, and producing it towards the base.

13. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

Proposition 22. Problem.

To describe a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given straight lines, of which any two are together greater than the third.

It is required to describe a triangle of which the sides shall be equal to A, B, C.

Construction. Take a straight line DE terminated at the point D, but unlimited towards E.

Make DF equal to A, FG equal to B, and GH equal to C. I. 3.

From centre F, with radius FD, describe the circle DLK.

From centre G with radius GH, describe the circle MHK, cutting the former circle at K.

Join FK, GK.

Then shall the triangle KFG have its sides equal to the three straight lines A, B, C.

Proof. Because F is the centre of the circle DLK,

therefore FK is equal to FD : *Def.* 11.

but FD is equal to A ; *Constr.*

therefore also FK is equal to A. *Ax.* 1.

Again, because G is the centre of the circle MHK,

therefore GK is equal to GH : *Def.* 11.

but GH is equal to C ; *Constr.*

therefore also GK is equal to C. *Ax.* 1.

And FG is equal to B. *Constr.*

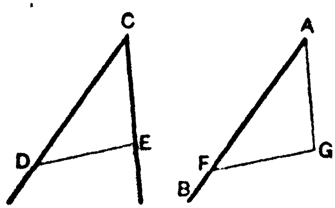
Therefore the triangle KFG has its sides KF, FG, GK equal respectively to the three given lines A, B, C. Q.E.F.

EXERCISE.

On a given base describe a triangle, whose remaining sides shall be equal to two given straight lines. Point out how the construction fails, if any one of the three given lines is greater than the sum of the other two.

Proposition 23. Problem.

At a given point in a given straight line, to make an angle equal to a given angle.



Let AB be the given straight line, and A the given point in it; and let DCE be the given angle.

It is required to draw from A a straight line making with AB an angle equal to the given angle DCE.

Construction. In CD, CE take any points D and E; and join DE.

From AB cut off AF equal to CD. I. 3.

On AF describe the triangle FAG, having the remaining sides AG, GF equal respectively to CE, ED. I. 22.

Then shall the angle FAG be equal to the angle DCE.

Proof. For in the triangles FAG, DCE,

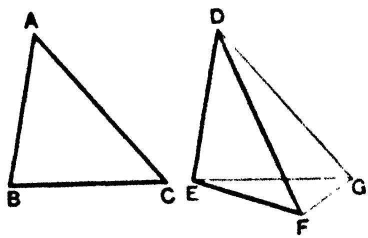
Because { FA is equal to DC, *Constr.*
and AG is equal to CE; *Constr.*
and the base FG is equal to the base DE: *Constr.* }

therefore the angle FAG is equal to the angle DCE. I. 8.

That is, AG makes with AB, at the given point A, an angle equal to the given angle DCE. Q.E.F.

PROPOSITION 24.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one greater than the angle contained by the corresponding sides of the other; then the base of that which has the greater angle shall be greater than the base of the other.



Let ABC, DEF be two triangles, in which the two sides BA, AC are equal to the two sides ED, DF, each to each, but the angle BAC greater than the angle EDF:

then shall the base BC be greater than the base EF.

* Of the two sides DE, DF, let DE be that which is not greater than DF.

Construction. At the point D, in the straight line ED, and on the same side of it as DF, make the angle EDG equal to the angle BAC. I. 23.

Make DG equal to DF or AC; I. 3.

and join EG, GF.

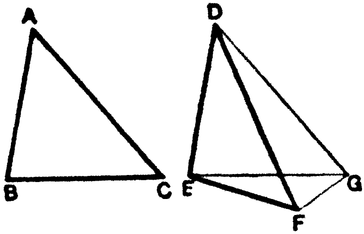
Proof. Then in the triangles BAC, EDG,

Because { BA is equal to ED, *Hyp.*
and AC is equal to DG, *Constr.*
also the contained angle BAC is equal to the contained angle EDG; *Constr.* }

Therefore the triangle BAC is equal to the triangle EDG in all respects: I. 4.

so that the base BC is equal to the base EG.

* See note on the next page.



Again, in the triangle FDG,
because DG is equal to DF,
therefore the angle DFG is equal to the angle DGF, I. 5.
but the angle DGF is greater than the angle EGF;
therefore also the angle DFG is greater than the angle EGF;
still more then is the angle EFG greater than the angle EGF.

And in the triangle EFG,
because the angle EFG is greater than the angle EGF,
therefore the side EG is greater than the side EF; I. 19.
but EG was shewn to be equal to BC;
therefore BC is greater than EF. Q.E.D.

* This condition was inserted by Simson to ensure that, in the complete construction, the point F should fall *below* EG. Without this condition it would be necessary to consider three cases: for F might fall *above*, or *upon*, or *below* EG; and each figure would require separate proof.

We are however scarcely at liberty to employ Simson's condition without *proving* that it fulfils the object for which it was introduced.

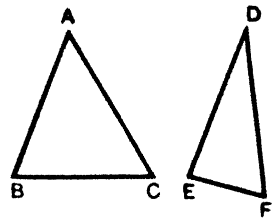
This may be done as follows:

Let EG, DF, produced if necessary, intersect at K.
Then, since DE is not greater than DF,
that is, since DE is not greater than DG,
therefore the angle DGE is not greater than the angle DEG. I. 18.
But the exterior angle DKG is greater than the angle DEK: I. 16.
therefore the angle DKG is greater than the angle DGK.
Hence DG is greater than DK. I. 19.
But DG is equal to DF;
therefore DF is greater than DK.
So that the point F must fall below EG.

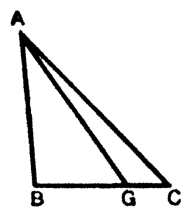
Or the following method may be adopted.

PROPOSITION 24. [ALTERNATIVE PROOF.]

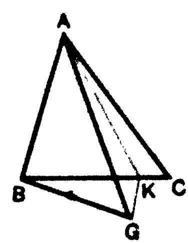
In the triangles ABC, DEF,
let BA be equal to ED,
and AC equal to DF,
but let the angle BAC be greater than the angle EDF:
then shall the base BC be greater than the base EF.



For apply the triangle DEF to the triangle ABC, so that D may fall on A, and DE along AB:
then because DE is equal to AB,
therefore E must fall on B.
And because the angle EDF is less than the angle BAC, *Hyp.*
therefore DF must fall between AB and AC.
Let DF occupy the position AG.



CASE I. If G falls on BC:
Then G must be between B and C:
therefore BC is greater than BG.
But BG is equal to EF:
therefore BC is greater than EF.



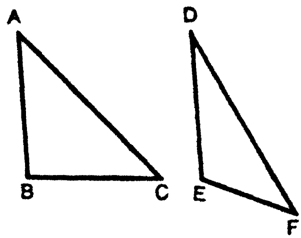
CASE II. If G does not fall on BC.
Bisect the angle CAG by the straight line AK which meets BC in K. I. 9.
Join GK.
Then in the triangles GAK, CAK,

Because $\left\{\begin{array}{l}\text{GA is equal to CA,} \quad \textit{Hyp.}\\ \text{and AK is common to both;}\\ \text{and the angle GAK is equal to the angle CAK;} \quad \textit{Constr.}\end{array}\right.$

therefore GK is equal to CK. I. 4.
But in the triangle BKG,
the two sides BK, KG are together greater than the third side BG, I. 20.
that is, BK, KC are together greater than BG;
therefore BC is greater than BG, or EF. Q.E.D.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; then the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the corresponding sides of the other.



Let ABC, DEF be two triangles which have the two sides BA, AC equal to the two sides ED, DF, each to each, but the base BC greater than the base EF:
then shall the angle BAC be greater than the angle EDF.

Proof. For if the angle BAC be not greater than the angle EDF, it must be either equal to, or less than the angle EDF.

But the angle BAC is not equal to the angle EDF,
for then the base BC would be equal to the base EF; I. 4.
but it is not. *Hyp.*

Neither is the angle BAC less than the angle EDF,
for then the base BC would be less than the base EF; I. 24.
but it is not. *Hyp.*

Therefore the angle BAC is neither equal to, nor less than the angle EDF;
that is, the angle BAC is greater than the angle EDF. Q.E.D.

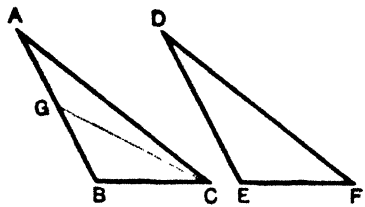
EXERCISE.

In a triangle ABC, the vertex A is joined to X, the middle point of the base BC; shew that the angle AXB is obtuse or acute, according as AB is greater or less than AC.

PROPOSITION 26. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and a side of one equal to a side of the other, these sides being either adjacent to the equal angles, or opposite to equal angles in each; then shall the triangles be equal in all respects.

CASE I. When the equal sides are *adjacent* to the equal angles in the two triangles.



Let ABC, DEF be two triangles, which have the angles ABC, ACB, equal to the two angles DEF, DFE, each to each; and the side BC equal to the side EF:
then shall the triangle ABC be equal to the triangle DEF in all respects;

that is, AB shall be equal to DE, and AC to DF, and the angle BAC shall be equal to the angle EDF.

For if AB be not equal to DE, one must be greater than the other. If possible, let AB be greater than DE.

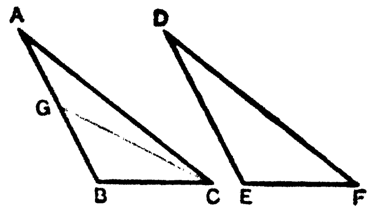
Construction. From BA cut off BG equal to ED, I. 3.
and join GC.

Proof. Then in the two triangles GBC, DEF,

Because { GB is equal to DE, *Constr.*
and BC to EF, *Hyp.*
also the contained angle GBC is equal to the contained angle DEF; *Hyp.* }

therefore the triangles are equal in all respects; I. 4.
so that the angle GCB is equal to the angle DFE.

But the angle ACB is equal to the angle DFE; *Hyp.*
therefore also the angle GCB is equal to the angle ACB; *Ax.* 1.
the part equal to the whole, which is impossible.



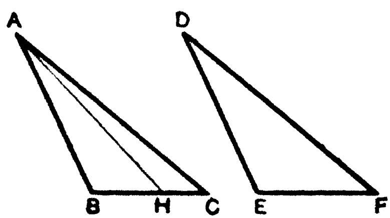
Therefore AB is not unequal to DE,
that is, AB is equal to DE.

Hence in the triangles ABC, DEF,

Because $\begin{cases} \text{AB is equal to DE,} & \textit{Proved.} \\ \text{and BC is equal to EF ;} & \textit{Hyp.} \\ \text{also the contained angle ABC is equal to the contained angle DEF ;} & \textit{Hyp.} \end{cases}$

therefore the triangles are equal in all respects : I. 4.
so that the side AC is equal to the side DF ;
and the angle BAC to the angle EDF. Q.E.D.

CASE II. When the equal sides are *opposite* to equal angles in the two triangles.



Let ABC, DEF be two triangles which have the angles ABC, ACB equal to the angles DEF, DFE, each to each, and the side AB equal to the side DE :

then shall the triangles ABC, DEF be equal in all respects ;
that is, BC shall be equal to EF, and AC to DF,
and the angle BAC shall be equal to the angle EDF.

For if BC be not equal to EF, one must be greater than the other. If possible, let BC be greater than EF.

Construction. From BC cut off BH equal to EF, I. 3.
and join AH.

Proof. Then in the triangles ABH, DEF,

Because { AB is equal to DE, *Hyp.*
and BH to EF, *Constr.*
also the contained angle ABH is equal to the contained angle DEF; *Hyp.*

therefore the triangles are equal in all respects, I. 4.
so that the angle AHB is equal to the angle DFE.

But the angle DFE is equal to the angle ACB; *Hyp.*
therefore the angle AHB is equal to the angle ACB; *Ax.* 1.
that is, an exterior angle of the triangle ACH is equal to an interior opposite angle; which is impossible. I. 16.

Therefore BC is not unequal to EF,
that is, BC is equal to EF.

Hence in the triangles ABC, DEF,

Because { AB is equal to DE, *Hyp.*
and BC is equal to EF; *Proved.*
also the contained angle ABC is equal to the contained angle DEF; *Hyp.*

therefore the triangles are equal in all respects; I. 4.
so that the side AC is equal to the side DF,
and the angle BAC to the angle EDF.

Q.E.D.

COROLLARY. *In both cases of this Proposition it is seen that the triangles may be made* to coincide with one another; *and they are therefore equal* in area.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

At the close of the first section of Book I., it is worth while to call special attention to those Propositions (viz. Props. 4, 8, 26) which deal with the *identical equality* of two triangles.

The results of these Propositions may be summarized thus:

Two triangles are equal to one another in all respects, when the following parts in each are equal, each to each.

1. Two sides, and the included angle. *Prop.* 4.
2. The three sides. *Prop.* 8, *Cor.*
3. (*a*) Two angles, and the adjacent side.
   (*b*) Two angles, and the side opposite one of them. } *Prop.* 26.

From this the beginner will perhaps surmise that two triangles may be shewn to be equal in all respects, when they have *three parts* equal, each to each; but to this statement two obvious exceptions must be made.

(i) When in two triangles the *three angles* of one are equal to the *three angles* of the other, each to each, it does *not* necessarily follow that the triangles are equal in all respects.

(ii) When in two triangles two sides of the one are equal to two sides of the other, each to each, and one angle equal to one angle, these not being the angles included by the equal sides; the triangles are *not* necessarily equal in all respects.

In these cases a further condition must be added to the hypothesis, before we can assert the identical equality of the two triangles.

[See Theorems and Exercises on Book I., Ex. 13, Page 92.]

We observe that in each of the three cases already of identical equality in two triangles, namely in Propositions 4, 8, 26, it is shewn that the triangles may be made to *coincide with one another*; so that they are equal in *area*, as in all other respects. Euclid however restricted himself to the use of Prop. 4, when he required to deduce the equality in *area* of two triangles from the equality of certain of their parts.

This restriction has been abandoned in the present text-book. [See note to Prop. 34.]

EXERCISES ON PROPOSITIONS 12—26.

1. If BX and CY, the bisectors of the angles at the base BC of an isosceles triangle ABC, meet the opposite sides in X and Y, shew that the triangles YBC, XCB are equal in all respects.

2. Shew that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

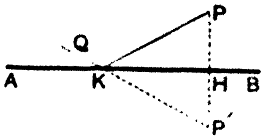
3. *Any point on the bisector of an angle is equidistant from the arms of the angle.*

4. Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B: shew that AX is equal to BY.

5. If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

6. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.*

7. *From two given points on the same side of a given straight line, draw two straight lines, which shall meet in the given straight line and make equal angles with it.*



Let AB be the given straight line, and P, Q the given points.

It is required to draw from P and Q to a point in AB, two straight lines that shall be equally inclined to AB.

Construction. From P draw PH perpendicular to AB: produce PH to P′, making HP′ equal to PH. Draw QP′, meeting AB in K. Join PK.

Then PK, QK shall be the required lines. [Supply the proof.]

8. In a given straight line find a point which is equidistant from two given intersecting straight lines. In what case is this impossible?

9. Through a given point draw a straight line such that the perpendiculars drawn to it from two given points may be equal.

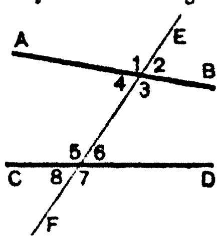
In what case is this impossible?

SECTION II.

PARALLEL STRAIGHT LINES AND PARALLELOGRAMS.

DEFINITION. Parallel straight lines are such as, being in the same plane, do not meet however far they are produced in both directions.

When two straight lines AB, CD are met by a third straight line EF, *eight* angles are formed, to which for the sake of distinction particular names are given.



Thus in the adjoining figure,

1, 2, 7, 8 are called **exterior** angles,

3, 4, 5, 6 are called **interior** angles,

4 and 6 are said to be **alternate** angles; so also the angles 3 and 5 are alternate to one another.

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the **interior opposite** angle on the same side of EF.

2 and 6 are sometimes called **corresponding** angles.

So also, 1 and 5, 7 and 3, 8 and 4 are corresponding angles.

Euclid's treatment of parallel straight lines is based upon his twelfth Axiom, which we here repeat.

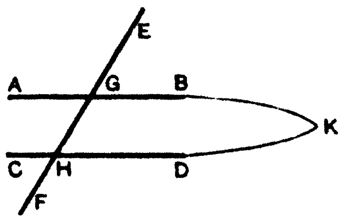
AXIOM 12. **If a straight line cut two straight lines so as to make the two interior angles on the same side of it together less than two right angles, these straight lines, being continually produced, will at length meet on that side on which are the angles which are together less than two right angles.**

Thus in the figure given above, if the two angles 3 and 6 are together less than two right angles, it is asserted that AB and CD will meet towards B and D.

This Axiom is used to establish I. 29: some remarks upon it will be found in a note on that Proposition.

PROPOSITION 27. THEOREM.

If a straight line, falling on two other straight lines, make the alternate angles equal to one another, then the straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD at G and H, so as to make the alternate angles AGH, GHD equal to one another:

then shall AB and CD be parallel.

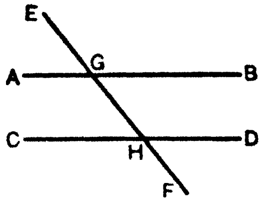
Proof. For if AB and CD be not parallel,
they will meet, if produced, either towards B and D, or towards A and C.
If possible, let AB and CD, when produced, meet towards B and D, at the point K.
Then KGH is a triangle, of which one side KG is produced to A:
therefore the exterior angle AGH is greater than the interior opposite angle GHK. I. 16.

But the angle AGH is equal to the angle GHK: *Hyp.*
hence the angles AGH and GHK are both equal and unequal;
which is impossible.
Therefore AB and CD cannot meet when produced towards B and D.
Similarly it may be shewn that they cannot meet towards A and C:

therefore they are parallel. Q. E. D.

PROPOSITION 28. THEOREM.

If a straight line, falling on two other straight lines, make an exterior angle equal to the interior opposite angle on the same side of the line; or if it make the interior angles on the same side together equal to two right angles, then the two straight lines shall be parallel.



Let the straight line EF cut the two straight lines AB, CD in G and H: and

First, let the exterior angle EGB be equal to the interior opposite angle GHD:

then shall AB and CD be parallel.

Proof. Because the angle EGB is equal to the angle GHD; and because the angle EGB is also equal to the vertically opposite angle AGH; I. 15.

therefore the angle AGH is equal to the angle GHD;

but these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q. E. D.

Secondly, let the two interior angles BGH, GHD be together equal to two right angles:

then shall AB and CD be parallel.

Proof. Because the angles BGH, GHD are together equal to two right angles; *Hyp.*

and because the adjacent angles BGH, AGH are also together equal to two right angles; I. 13.

therefore the angles BGH, AGH are together equal to the two angles BGH, GHD.

From these equals take the common angle BGH:

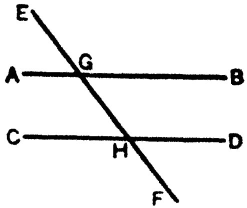
then the remaining angle AGH is equal to the remaining angle GHD: and these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q. E. D.

Proposition 29. Theorem.

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal to one another, and the exterior angle equal to the interior opposite angle on the same side; and also the two interior angles on the same side equal to two right angles.



Let the straight line EF fall on the parallel straight lines AB, CD:

then (i) the alternate angles AGH, GHD shall be equal to one another;

(ii) the exterior angle EGB shall be equal to the interior opposite angle GHD;

(iii) the two interior angles BGH, GHD shall be together equal to two right angles.

Proof. (i) For if the angle AGH be not equal to the angle GHD, one of them must be greater than the other.

If possible, let the angle AGH be greater than the angle GHD;

add to each the angle BGH:

then the angles AGH, BGH are together greater than the angles BGH, GHD.

But the adjacent angles AGH, BGH are together equal to two right angles; I. 13.

therefore the angles BGH, GHD are together less than two right angles;

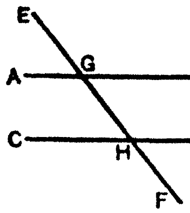
therefore AB and CD meet towards B and D. *Ax.* 12.

But they never meet, since they are parallel. *Hyp.*

Therefore the angle AGH is not unequal to the angle GHD:

that is, the alternate angles AGH, GHD are equal.

(Over)



(ii) Again, because the angle AGH is equal to the vertically opposite angle EGB; I. 15.
and because the angle AGH is equal to the angle GHD; *Proved.*
therefore the exterior angle EGB is equal to the interior opposite angle GHD.

(iii) Lastly, the angle EGB is equal to the angle GHD; *Proved.*
add to each the angle BGH;
then the angles EGB, BGH are together equal to the angles BGH, GHD.
But the adjacent angles EGB, BGH are together equal to two right angles: I. 13.
therefore also the two interior angles BGH, GHD are together equal to two right angles. Q. E. D.

EXERCISES ON PROPOSITIONS 27, 28, 29.

1. Two straight lines AB, CD bisect one another at O: shew that the straight lines joining AC and BD are parallel. [I. 27.]

2. *Straight lines which are perpendicular to the same straight line are parallel to one another.* [I. 27 or I. 28.]

3. *If a straight line meet two or more parallel straight lines, and is perpendicular to one of them, it is also perpendicular to all the others.* [I. 29.]

4. *If two straight lines are parallel to two other straight lines, each to each, then the angles contained by the first pair are equal respectively to the angles contained by the second pair.* [I. 29.]

Note on the Twelfth Axiom.

It must be admitted that Euclid's twelfth Axiom is unsatisfactory as the basis of a theory of parallel straight lines. It cannot be regarded as either simple or self-evident, and it therefore falls short of the essential characteristics of an axiom: nor is the difficulty entirely removed by considering it as a corollary to Proposition 17, of which it is the converse.

Many substitutes have been proposed; but we need only notice here the system which has met with most general approval.

This system rests on the following hypothesis, which is put forward as a fundamental Axiom.

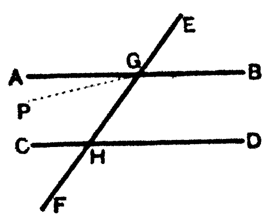
Axiom. *Two intersecting straight lines cannot be both parallel to a third straight line.*

This statement is known as **Playfair's Axiom**; and though it is not altogether free from objection, it is recommended as both simpler and more fundamental than that employed by Euclid, and more readily admitted without proof.

Propositions 27 and 28 having been proved in the usual way, the first part of Proposition 29 is then given thus.

Proposition 29. [Alternative Proof.]

If a straight line fall on two parallel straight lines, then it shall make the alternate angles equal.



Let the straight line EF meet the two parallel straight lines AB, CD, at G and H:

then shall the alternate angles AGH, GHD be equal.

For if the angle AGH is not equal to the angle GHD:

at G in the straight line HG make the angle HGP equal to the angle GHD, and alternate to it. I. 23.

Then PG and CD are parallel. I. 27.

But AB and CD are parallel: *Hyp.*

therefore the two intersecting straight lines AG, PG are both parallel to CD:

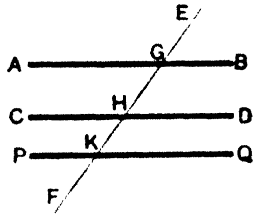
which is impossible. *Playfair's Axiom.*

Therefore the angle AGH is not unequal to the angle GHD, that is, the alternate angles AGH, GHD are equal. Q.E.D.

The second and third parts of the Proposition may then be deduced as in the text; and Euclid's Axiom 12 follows as a Corollary.

PROPOSITION 30. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ:

then shall AB and CD be parallel to one another.

Construction. Draw any straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them,

therefore the angle AGK is equal to the alternate angle GKQ. I. 29.

And because CD and PQ are parallel, and EF meets them, therefore the exterior angle GHD is equal to the interior opposite angle HKQ. I. 29.

Therefore the angle AGH is equal to the angle GHD;

and these are alternate angles;

therefore AB and CD are parallel. I. 27.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition may be established in a similar manner, though in this case it scarcely needs proof; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

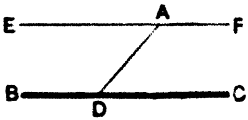
The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line: which is impossible.

Therefore AB and CD never meet; that is, they are parallel.

PROPOSITION 31. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given point, and BC the given straight line. It is required to draw through A a straight line parallel to BC.

Construction. In BC take any point D; and join AD. At the point A in DA, make the angle DAE equal to the angle ADC, and alternate to it I. 23.

and produce EA to F.

Then shall EF be parallel to BC.

Proof. Because the straight line AD, meeting the two straight lines EF, BC, makes the alternate angles EAD, ADC equal; *Constr.*

therefore EF is parallel to BC; I. 27.

and it has been drawn through the given point A.

Q. E. F.

EXERCISES.

1. Any straight line drawn parallel to the base of an isosceles triangle makes equal angles with the sides.

2. If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles.

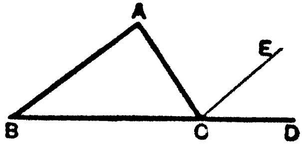
3. From a given point draw a straight line that shall make with a given straight line an angle equal to a given angle.

4. From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z: shew the triangle AYZ is isosceles.

5. If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, shew that the triangle is isosceles.

Proposition 32. Theorem.

If a side of a triangle be produced, then the exterior angle shall be equal to the sum of the two interior opposite angles: also the three interior angles of a triangle are together equal to two right angles.



Let ABC be a triangle, and let one of its sides BC be produced to D:

then (i) the exterior angle ACD shall be equal to the sum of the two interior opposite angles CAB, ABC;

(ii) the three interior angles ABC, BCA, CAB shall be together equal to two right angles.

Construction. Through C draw CE parallel to BA. I. 31.

Proof. (i) Then because BA and CE are parallel, and AC meets them,

therefore the angle ACE is equal to the alternate angle CAB. I. 29.

Again, because BA and CE are parallel, and BD meets them,

therefore the exterior angle ECD is equal to the interior opposite angle ABC. I. 29.

Therefore the whole exterior angle ACD is equal to the sum of the two interior opposite angles CAB, ABC.

(ii) Again, since the angle ACD is equal to the sum of the angles CAB, ABC; *Proved.*

to each of these equals add the angle BCA:

then the angles BCA, ACD are together equal to the three angles BCA, CAB, ABC.

But the adjacent angles BCA, ACD are together equal to two right angles; I. 13.

therefore also the angles BCA, CAB, ABC are together equal to two right angles. Q. E. D.

From this Proposition we draw the following important inferences.

1. *If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angle of the one is equal to the third angle of the other.*

2. *In any right-angled triangle the two acute angles are complementary.*

3. *In a right-angled isosceles triangle each of the equal angles is half a right angle.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

6. *Each angle of an equilateral triangle is two-thirds of a right angle.*

EXERCISES ON PROPOSITION 32

1. Prove that the three angles of a triangle are together equal to two right angles,

   (i) by drawing through the vertex a straight line parallel to the base;

   (ii) by joining the vertex to any point in the base.

2. If the base of any triangle is produced both ways, shew that the sum of the two exterior angles diminished by the vertical angle is equal to two right angles.

3. *If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair.*

4. *Every right-angled triangle is divided into two isosceles triangles by a straight line drawn from the right angle to the middle point of the hypotenuse.*

Hence the joining line is equal to half the hypotenuse.

5. *Draw a straight line at right angles to a given finite straight line from one of its extremities, without producing the given straight line.*

[Let AB be the given straight line. On AB describe any isosceles triangle ACB. Produce BC to D, making CD equal to BC. Join AD. Then shall AD be perpendicular to AB.]

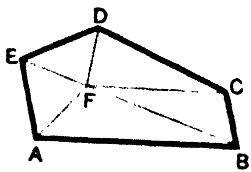
6. *Trisect a right angle.*

7. The angle contained by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle formed by producing the base.

8. The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles.

The following theorems were added as corollaries to Proposition 32 by Robert Simson.

COROLLARY 1. *All the interior angles of any rectilineal figure, with four right angles, are together equal to twice as many right angles as the figure has sides.*



Let ABCDE be any rectilineal figure.

Take F, any point within it,

and join F to each of the angular points of the figure.

Then the figure is divided into as many triangles as it has sides.

And the three angles of each triangle are together equal to two right angles. I. 32.

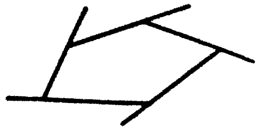
Hence all the angles of all the triangles are together equal to twice as many right angles as the figure has sides.

But all the angles of all the triangles make up the interior angles of the figure, together with the angles at F;

and the angles at F are together equal to four right angles: I. 15, *Cor.*

Therefore all the interior angles of the figure, with four right angles, are together equal to twice as many right angles as the figure has sides. Q. E. D.

COROLLARY 2. *If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*



For at each angular point of the figure, the interior angle and the exterior angle are together equal to two right angles. I. 13.

Therefore all the interior angles, with all the exterior angles, are together equal to twice as many right angles as the figure has sides.

But all the interior angles, with four right angles, are together equal to twice as many right angles as the figure has sides. I. 32, *Cor* 1.

Therefore all the interior angles, with all the exterior angles, are together equal to all the interior angles, with four right angles.

Therefore the exterior angles are together equal to four right angles. Q. E. D.

EXERCISES ON SIMSON'S COROLLARIES.

[A polygon is said to be **regular** when it has all its sides and all its angles equal.]

1. Express in terms of a right angle the magnitude of each angle of (i) a regular hexagon, (ii) a regular octagon.

2. If one side of a regular hexagon is produced, shew that the exterior angle is equal to the angle of an equilateral triangle.

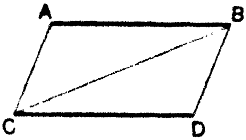
3. Prove Simson's first Corollary by joining one vertex of the rectilineal figure to each of the other vertices.

4. Find the magnitude of each angle of a regular polygon of n sides.

5. If the alternate sides of any polygon be produced to meet, the sum of the included angles, together with eight right angles, will be equal to twice as many right angles as the figure has sides.

Proposition 33. Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD:

then shall AC and BD be equal and parallel.

Construction. Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

therefore the alternate angles ABC, BCD are equal. I. 29.

Now in the triangles ABC, DCB,

Because {
AB is equal to DC, *Hyp.*
and BC is common to both;
also the angle ABC is equal to the angle DCB; *Proved.*
}

therefore the triangles are equal in all respects; I. 4.

so that the base AC is equal to the base DB,

and the angle ACB equal to the angle DBC;

but these are alternate angles;

therefore AC and BD are parallel: I. 27

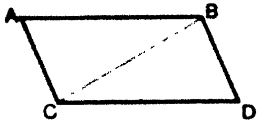
and it has been shewn that they are also equal.

Q. E. D.

Definition. A **Parallelogram** is a four-sided figure whose opposite sides are parallel.

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ACDB be a parallelogram, of which BC is a diagonal: then shall the opposite sides and angles of the figure be equal to one another; and the diagonal BC shall bisect it.

Proof. Because AB and CD are parallel, and BC meets them,

therefore the alternate angles ABC, DCB are equal. I. 29.

Again, because AC and BD are parallel, and BC meets them,

therefore the alternate angles ACB, DBC are equal. I. 29.

Hence in the triangles ABC, DCB,

Because { the angle ABC is equal to the angle DCB, and the angle ACB is equal to the angle DBC; also the side BC, which is adjacent to the equal angles, is common to both,

therefore the two triangles ABC, DCB are equal in all respects; I. 26.

so that AB is equal to DC, and AC to DB;

and the angle BAC is equal to the angle CDB.

Also, because the angle ABC is equal to the angle DCB,

and the angle CBD equal to the angle BCA,

therefore the whole angle ABD is equal to the whole angle DCA.

And since it has been shewn that the triangles ABC, DCB are equal in all respects,

therefore the diagonal BC bisects the parallelogram ACDB.

Q. E. D.

[See note on next page.]

NOTE. To the proof which is here given Euclid added an application of Proposition 4, with a view to shewing that the triangles ABC, DCB are equal *in area*, and that therefore the diagonal BC bisects the parallelogram. This equality of area is however sufficiently established by the step which depends upon I. 26. [See page 48.]

EXERCISES.

1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

2. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

3. *If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

4. *If a quadrilateral has all its sides equal and one angle a right angle, all its angles are right angles.*

5. *The diagonals of a parallelogram bisect each other.*

6. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

7. If two opposite angles of a parallelogram are bisected by the diagonal which joins them, the figure is equilateral.

8. If the diagonals of a parallelogram are equal, all its angles are right angles.

9. In a parallelogram which is not rectangular the diagonals are unequal.

10. Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point.

11. *If two parallelograms have two adjacent sides of one equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other, the parallelograms are equal in all respects.*

12. *Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each.*

13. In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal.

14. If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC; shew that the figure AYCX is a parallelogram.

MISCELLANEOUS EXERCISES ON SECTIONS I. AND II.

1. Shew that the construction in Proposition 2 may generally be performed in eight different ways. Point out the exceptional case.

2. The bisectors of two vertically opposite angles are in the same straight line.

3. In the figure of Proposition 16, if AF is joined, shew

(i) that AF is equal to BC;

(ii) that the triangle ABC is equal to the triangle CFA in all respects.

4. ABC is a triangle right angled at B, and BC is produced to D: shew that the angle ACD is obtuse.

5. Shew that in any regular polygon of n sides each angle contains $\frac{2(n-2)}{n}$ right angles.

6. The angle contained by the bisectors of the angles at the base of any triangle is equal to the vertical angle together with half the sum of the base angles.

7. The angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two corresponding interior angles.

8. If perpendiculars are drawn to two intersecting straight lines from any point between them, shew that the bisector of the angle between the perpendiculars is parallel to (or coincident with) the bisector of the angle between the given straight lines.

9. If two points P, Q be taken in the equal sides of an isosceles triangle ABC, so that BP is equal to CQ, shew that PQ is parallel to BC.

10. ABC and DEF are two triangles, such that AB, BC are equal and parallel to DE, EF, each to each; shew that AC is equal and parallel to DF.

11. Prove the second Corollary to Prop. 32 by drawing through any angular point lines parallel to all the sides.

12. If two sides of a quadrilateral are parallel, and the remaining two sides equal but not parallel, shew that the opposite angles are supplementary; also that the diagonals are equal.

SECTION III.

THE AREAS OF PARALLELOGRAMS AND TRIANGLES.

Hitherto when two figures have been said to be *equal*, it has been implied that they are *identically* equal, that is, equal in all

In Section III. of Euclid's first Book, we have to consider the equality in *area* of parallelograms and triangles which are not necessarily equal in all :

[The ultimate test of equality, as we have already seen, is afforded by Axiom 8, which asserts that magnitudes which *may be made to coincide with one another* are equal. Now figures which are not identically equal, cannot be made to coincide without first undergoing some change of form: hence the method of direct *superposition* is unsuited to the purposes of the present section.

We shall see however from Euclid's proof of Proposition 35, that two figures which are not identically equal, may nevertheless be so related to a third figure, that it is possible to infer the equality of their areas.]

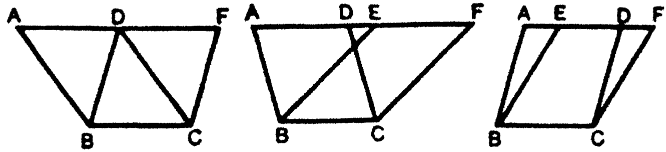
DEFINITIONS.

1. The **Altitude** of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side.

2. The **Altitude** of a triangle with reference to a given side as base, is the perpendicular distance of the vertex from the

PROPOSITION 35. THEOREM.

Parallelograms on the same base, and between the same parallels, are equal in area.



Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels BC, AF:

then shall the parallelogram ABCD be equal in area to the parallelogram EBCF.

CASE I. If the sides of the given parallelograms, opposite to the common base BC, are terminated at the same point D:

then because each of the parallelograms is double of the triangle BDC; I. 34.

therefore they are equal to one another. *Ax.* 6.

CASE II. But if the sides AD, EF, opposite to the base BC, are not terminated at the same point:

then because ABCD is a parallelogram,

therefore AD is equal to the opposite side BC; I. 34

and for a similar reason, EF is equal to BC;

therefore AD is equal to EF. *Ax.* 1.

Hence the whole, or remainder, EA is equal to the whole, or remainder, FD.

Then in the triangles FDC, EAB,

Because { FD is equal to EA, *Proved.*
and DC is equal to the opposite side AB, I. 34.
also the exterior angle FDC is equal to the interior opposite angle EAB, I. 29.

therefore the triangle FDC is equal to the triangle EAB. I. 4.

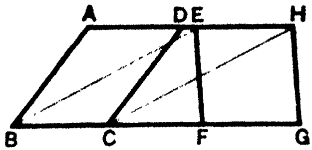
From the whole figure ABCF take the triangle FDC; and from the same figure take the equal triangle EAB;

then the remainders are equal; *Ax.* 3.

that is, the parallelogram ABCD is equal to the parallelogram EBCF. Q. E. D.

Proposition 36. Theorem.

Parallelograms on equal bases, and between the same parallels, are equal in area.



Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG:
then shall the parallelogram ABCD be equal to the parallelogram EFGH.

Construction. Join BE, CH.

Proof. Then because BC is equal to FG; *Hyp.*
and FG is equal to the opposite side EH; I. 34.
therefore BC is equal to EH: *Ax.* 1.
and they are also parallel; *Hyp.*
therefore BE and CH, which join them towards the same parts, are also equal and parallel. I. 33.
Therefore EBCH is a parallelogram. *Def.* 26.

Now the parallelogram ABCD is equal to EBCH;
for they are on the same base BC, and between the same parallels BC, AH. I. 35.

Also the parallelogram EFGH is equal to EBCH;
for they are on the same base EH, and between the same parallels EH, BG. I. 35.

Therefore the parallelogram ABCD is equal to the parallelogram EFGH. *Ax.* 1.

Q. E. D.

From the last two Propositions we infer that:

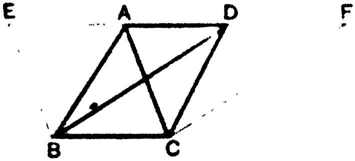
(i) *A parallelogram is equal in area to a rectangle of equal base and equal altitude.*

(ii) *Parallelograms on equal bases and of equal altitudes are equal in area.*

(iii) *Of two parallelograms of equal altitudes, that is the greater which has the greater base; and of two parallelograms on equal bases, that is the greater which has the greater altitude.*

Proposition 37. Theorem.

Triangles on the same base, and between the same parallels, are equal in area.



Let the triangles ABC, DBC be upon the same base BC, and between the same parallels BC, AD.

Then shall the triangle ABC be equal to the triangle DBC.

Construction. Through B draw BE parallel to CA, to meet DA produced in E; I. 31.

through C draw CF parallel to BD, to meet AD produced in F.

Proof. Then, by construction, each of the figures EBCA, DBCF is a parallelogram. *Def.* 26.

And EBCA is equal to DBCF;

for they are on the same base BC, and between the same parallels BC, EF. I. 35.

And the triangle ABC is half of the parallelogram EBCA, for the diagonal AB bisects it. I. 34.

Also the triangle DBC is half of the parallelogram DBCF, for the diagonal DC bisects it. I. 34.

But the halves of equal things are equal; *Ax.* 7.

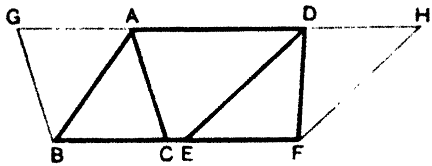
therefore the triangle ABC is equal to the triangle DBC.

Q.E.D.

[For Exercises see page 73.]

PROPOSITION 38. THEOREM.

Triangles on equal bases, and between the same parallels, are equal in area.



Let the triangles ABC, DEF be on equal bases BC, EF, and between the same parallels BF, AD:

then shall the triangle ABC be equal to the triangle DEF.

Construction. Through B draw BG parallel to CA, to meet DA produced in G; I. 31.

through F draw FH parallel to ED, to meet AD produced in H.

Proof. Then, by construction, each of the figures GBCA, DEFH is a parallelogram. *Def.* 26.

And GBCA is equal to DEFH;

for they are on equal bases BC, EF, and between the same parallels BF, GH. I. 36.

And the triangle ABC is half of the parallelogram GBCA, for the diagonal AB bisects it. I. 34.

Also the triangle DEF is half the parallelogram DEFH, for the diagonal DF bisects it. I. 34.

But the halves of equal things are equal: *Ax.* 7.

therefore the triangle ABC is equal to the triangle DEF.

Q.E.D.

From this Proposition we infer that:

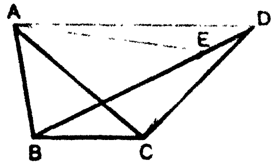
(i) *Triangles on equal bases and of equal altitude are equal in area.*

(ii) *Of two triangles of the same altitude, that is the greater which has the greater base: and of two triangles on the same base, or on equal bases, that is the greater which has the greater altitude.*

[For Exercises see page 73.]

PROPOSITION 39. THEOREM.

Equal triangles on the same base, and on the same side of it, are between the same parallels.



Let the triangles ABC, DBC which stand on the same base BC, and on the same side of it, be equal in area.

then shall they be between the same parallels;

that is, if AD be joined, AD shall be parallel to BC.

Construction. For if AD be not parallel to BC,

if possible, through A draw AE parallel to BC, I. 31.

meeting BD, or BD produced, in E.

Join EC.

Proof. Now the triangle ABC is equal to the triangle EBC, for they are on the same base BC, and between the same parallels BC, AE. I. 37.

But the triangle ABC is equal to the triangle DBC; *Hyp.* therefore also the triangle DBC is equal to the triangle EBC; the whole equal to the part; which is impossible.

Therefore AE is not parallel to BC.

Similarly it can be shewn that no other straight line through A, except AD, is parallel to BC.

Therefore AD is parallel to BC.

Q.E.D.

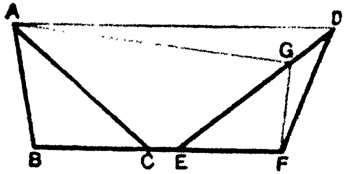
From this Proposition it follows that:

Equal triangles on the same base have equal altitudes.

[For Exercises see page 73.]

PROPOSITION 40. THEOREM.

Equal triangles, on equal bases in the same straight line, and on the same side of it, are between the same parallels.



Let the triangles ABC, DEF which stand on equal bases BC, EF, in the same straight line BF, and on the same side of it, be equal in area:

then shall they be between the same parallels;

that is, if AD be joined, AD shall be parallel to BF.

Construction. For if AD be not parallel to BF,

if possible, through A draw AG parallel to BF, I. 31.

meeting ED, or ED produced, in G.

Join GF.

Proof. Now the triangle ABC is equal to the triangle GEF, for they are on equal bases BC, EF, and between the same parallels BF, AG. I. 38.

But the triangle ABC is equal to the triangle DEF: *Hyp.*

therefore also the triangle DEF is equal to the triangle GEF:

the whole equal to the part; which is impossible.

Therefore AG is not parallel to BF.

Similarly it can be shown that no other straight line through A, except AD, is parallel to BF.

Therefore AD is parallel to BF.

Q.E.D.

From this Proposition it follows that:

(i) *Equal triangles on equal bases have equal altitudes.*

(ii) *Equal triangles of equal altitudes have equal bases.*

EXERCISES ON PROPOSITIONS 37—40.

DEFINITION. Each of the three straight lines which join the angular points of a triangle to the middle points of the opposite sides is called a **Median** of the triangle.

ON PROP. 37.

1. If, in the figure of Prop. 37, AC and BD intersect in K, shew that
   (i) the triangles AKB, DKC are equal in area.
   (ii) the quadrilaterals EBKA, FCKD are equal.

2. In the figure of I. 16, shew that the triangles ABC, FBC are equal in area.

3. On the base of a given triangle construct a second triangle, equal in area to the first, and having its vertex in a given straight line.

4. Describe an isosceles triangle equal in area to a given triangle and standing on the same base.

ON PROP. 38.

5. *A triangle is divided by each of its medians into two parts of equal area.*

6. A parallelogram is divided by its diagonals into four triangles of equal area.

7. ABC is a triangle, and its base BC is bisected at X; if Y be any point in the median AX, shew that the triangles ABY, ACY are equal in area.

8. In AC, a diagonal of the parallelogram ABCD, any point X is taken, and XB, XD are drawn: shew that the triangle BAX is equal to the triangle DAX.

9. If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides *supplementary*, the triangles are equal in area.

ON PROP. 39.

10. *The straight line which joins the middle points of two sides of a triangle is parallel to the third side.*

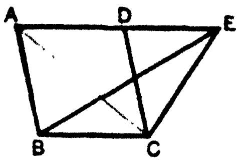
11. *If two straight lines* AB, CD *intersect in* O, *so that the triangle* AOC *is equal to the triangle* DOB, *shew that* AD *and* CB *are parallel.*

ON PROP. 40.

12. Deduce Prop. 40 from Prop. 39 by joining AE, AF in the figure of page 72.

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.



Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE:

then shall the parallelogram ABCD be double of the triangle EBC.

Construction. Join AC.

Proof. Then the triangle ABC is equal to the triangle EBC, for they are on the same base BC, and between the same parallels BC, AE. I. 37.

But the parallelogram ABCD is double of the triangle ABC, for the diagonal AC bisects the parallelogram. I. 34.

Therefore the parallelogram ABCD is also double of the triangle EBC. Q.E.D.

EXERCISES.

1. ABCD is a parallelogram, and X, Y are the middle points of the sides AD, BC; if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD.

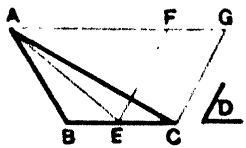
2. Describe a right-angled isosceles triangle equal to a given square.

3. If ABCD is a parallelogram, and XY any points in DC and AD respectively: shew that the triangles AXB, BYC are equal in area.

4. ABCD is a parallelogram, and P is any point within it; shew that the sum of the triangles PAB, PCD is equal to half the parallelogram.

PROPOSITION 42. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.



Let ABC be the given triangle, and D the given angle.
It is required to describe a parallelogram equal to ABC, and having one of its angles equal to D.

Construction. Bisect BC at E. I. 10.

At E in CE, make the angle CEF equal to D; I. 23.

through A draw AFG parallel to EC; I. 31.

and through C draw CG parallel to EF.

Then FECG shall be the parallelogram required.

Join AE.

Proof. Now the triangles ABE, AEC are equal,
for they are on equal bases BE, EC, and between the same parallels; I. 38.

therefore the triangle ABC is double of the triangle AEC.

But FECG is a parallelogram by construction: *Def.* 26.

and it is double of the triangle AEC,
for they are on the same base EC, and between the same parallels EC and AG. I. 41.

Therefore the parallelogram FECG is equal to the triangle ABC;
and it has one of its angles CEF equal to the given angle D.

Q. E. F.

EXERCISES.

1. Describe a parallelogram equal to a given square standing on the same base, and having an angle equal to half a right angle.

2. Describe a rhombus equal to a given parallelogram and standing on the same base. When does the construction fail?

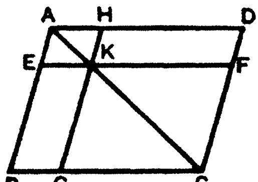
DEFINITION. If in the diagonal of a parallelogram any point is taken, and straight lines are drawn through it parallel to the sides of the parallelogram; then of the four parallelograms into which the whole figure is divided, the two through which the diagonal passes are called **Parallelograms about that diagonal**, and the other two, which with these make up the whole figure, are called the **complements** of the parallelograms about the diagonal.

Thus in the figure given below, AEKH, KGCF are parallelograms about the diagonal AC; and HKFD, EBGK are the complements of those parallelograms.

NOTE. A parallelogram is often named by *two* letters only, these being placed at opposite angular points.

PROPOSITION 43. THEOREM.

The complements of the parallelograms about the diagonal of any parallelogram, are equal to one another.



Let ABCD be a parallelogram, and KD, KB the complements of the parallelograms EH, GF about the diagonal AC: then shall the complement BK be equal to the complement KD.

Proof. Because EH is a parallelogram, and AK its diagonal, therefore the triangle AEK is equal to the triangle AHK. I. 34.

For a similar reason the triangle KGC is equal to the triangle KFC.

Hence the triangles AEK, KGC are together equal to the triangles AHK, KFC.

But the whole triangle ABC is equal to the whole triangle ADC, for AC bisects the parallelogram ABCD ; I. 34.
therefore the remainder, the complement BK, is equal to the remainder, the complement KD. Q.E.D.

EXERCISES.

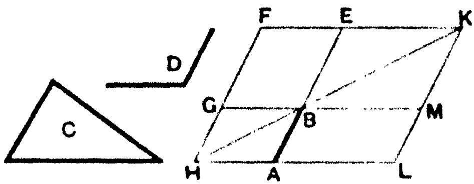
In the figure of Prop. 43, prove that

(i) The parallelogram ED is equal to the parallelogram BH.

(ii) If KB, KD are joined, the triangle AKB is equal to the triangle AKD.

PROPOSITION 44. PROBLEM.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.



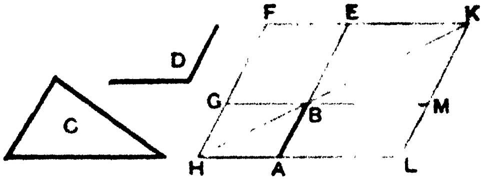
Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to the angle D.

Construction. On AB produced describe a parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D; I. 22 *and* I. 42*.
through A draw AH parallel to BG or EF, to meet FG produced in H. I. 31.

Join HB.

* This step of the construction is effected by first describing on AB produced a triangle whose sides are respectively equal to those of the triangle C (I. 22); and by then making a parallelogram equal to the triangle so drawn, and having an angle equal to D (I. 42).



Then because AH and EF are parallel, and HF meets them, therefore the angles AHF, HFE are together equal to two right angles: I. 29.

hence the angles BHF, HFE are together less than two right angles;

therefore HB and FE will meet if produced towards B and E. *Ax.* 12.

Produce them to meet at K.

Through K draw KL parallel to EA or FH; I. 31.

and produce HA, GB to meet KL in the points L and M.

Then shall BL be the parallelogram required.

Proof. Now FHLK is a parallelogram, *Constr.*

and LB, BF are the complements of the parallelograms about the diagonal HK:

therefore LB is equal to BF. I. 43.

But the triangle C is equal to BF; *Constr.*

therefore LB is equal to the triangle C.

And because the angle GBE is equal to the vertically opposite angle ABM, I. 15.

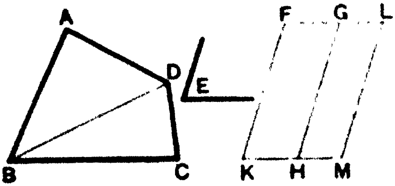
and is likewise equal to the angle D; *Constr.*

therefore the angle ABM is equal to the angle D.

Therefore the parallelogram LB, which is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Q.E.F.

Proposition 45. Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given angle.



Let ABCD be the given rectilineal figure, and E the given angle.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Suppose the given rectilineal figure to be a quadrilateral.

Construction. Join BD.

Describe the parallelogram FH equal to the triangle ABD, and having the angle FKH equal to the angle E. I. 42.

To GH apply the parallelogram GM, equal to the triangle DBC, and having the angle GHM equal to E. I. 44.

Then shall FKML be the parallelogram required.

Proof. Because each of the angles GHM, FKH is equal to E, therefore the angle FKH is equal to the angle GHM.

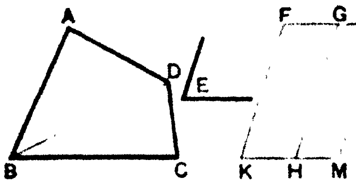
To each of these equals add the angle GHK;

then the angles FKH, GHK are together equal to the angles GHM, GHK.

But since FK, GH are parallel, and KH meets them, therefore the angles FKH, GHK are together equal to two right angles: I. 29.

therefore also the angles GHM, GHK are together equal to two right angles:

therefore KH, HM are in the same straight line. I. 14.



Again, because KM, FG are parallel, and HG meets them,
therefore the alternate angles MHG, HGF are equal : I. 29
to each of these equals add the angle HGL ;
then the angles MHG, HGL are together equal to the angles HGF, HGL.

But because HM, GL are parallel, and HG meets them,
therefore the angles MHG, HGL are together equal to two right angles: I. 29.
therefore also the angles HGF, HGL are together equal to two right angles ;
therefore FG, GL are in the same straight line. I. 14.

And because KF and ML are each parallel to HG, *Constr.*
therefore KF is parallel to ML ; I. 30.
and KM, FL are parallel ; *Constr.*
therefore FKML is a parallelogram. *Def.* 26.

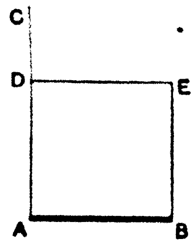
And because the parallelogram FH is equal to the triangle ABD, *Constr.*
and the parallelogram GM to the triangle DBC ; *Constr.*
therefore the whole parallelogram FKML is equal to the whole figure ABCD ;
and it has the angle FKM equal to the angle E.

By a series of similar steps, a parallelogram may be constructed equal to a rectilineal figure of more than four sides.

Q.E.F.

PROPOSITION 46. PROBLEM.

To describe a square on a given straight line.



Let AB be the given straight line:
it is required to describe a square on AB.

Constr. From A draw AC at right angles to AB; I. 11.
and make AD equal to AB. I. 3.
Through D draw DE parallel to AB; I. 31.
and through B draw BE parallel to AD, meeting DE in E.
Then shall ADEB be a square.

Proof. For, by construction, ADEB is a parallelogram:
therefore AB is equal to DE, and AD to BE. I. 34.
But AD is equal to AB; *Constr.*
therefore the four straight lines AB, AD, DE, EB are equal to one another;
that is, the figure ADEB is equilateral.

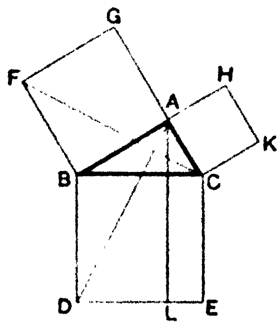
Again, since AB, DE are parallel, and AD meets them,
therefore the angles BAD, ADE are together equal to two right angles; I. 29.
but the angle BAD is a right angle; *Constr.*
therefore also the angle ADE is a right angle.
And the opposite angles of a parallelogram are equal; I. 34.
therefore each of the angles DEB, EBA is a right angle:
that is the figure ADEB is rectangular.
Hence it is a square, and it is described on AB.

Q. E. F.

COROLLARY. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

Proposition 47. Theorem.

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled triangle, having the angle BAC a right angle:
then shall the square described on the hypotenuse BC be equal to the sum of the squares described on BA, AC.

Construction. On BC describe the square BDEC; I. 46.
and on BA, AC describe the squares BAGF, ACKH.
Through A draw AL parallel to BD or CE; I. 31.
and join AD, FC.

Proof. Then because each of the angles BAC, BAG is a right angle,
therefore CA and AG are in the same straight line. I. 14.

Now the angle CBD is equal to the angle FBA,
for each of them is a right angle.
Add to each the angle ABC:
then the whole angle ABD is equal to the whole angle FBC.

Then in the triangles ABD, FBC,

Because { AB is equal to FB,
and BD is equal to BC,
also the angle ABD is equal to the angle FBC;

therefore the triangle ABD is equal to the triangle FBC. I. 4.

Now the parallelogram BL is double of the triangle ABD, for they are on the same base BD, and between the same parallels BD, AL. I. 41.

And the square GB is double of the triangle FBC, for they are on the same base FB, and between the same parallels FB, GC. I. 41.

But doubles of equals are equal: *Ax.* 6.

therefore the parallelogram BL is equal to the square GB.

In a similar way, by joining AE, BK, it can be shewn that the parallelogram CL is equal to the square CH.

Therefore the whole square BE is equal to the sum of the squares GB, HC:

that is, the square described on the hypotenuse BC is equal to the sum of the squares described on the two sides BA, AC. Q.E.D.

NOTE. It is not necessary to the proof of this Proposition that the three squares should be described *external* to the triangle ABC; and since *each* square may be drawn either *towards* or *away from* the triangle, it may be shewn that there are $2 \times 2 \times 2$, or *eight*, possible constructions.

EXERCISES.

1. In the figure of this Proposition, shew that

    (i) If BG, CH are joined, these straight lines are parallel;

    (ii) The points F, A, K are in one straight line;

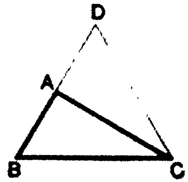
    (iii) FC and AD are at right angles to one another;

    (iv) If GH, KE, FD are joined, the triangle GAH is equal to the given triangle in all respects; and the triangles FBD, KCE are each equal in area to the triangle ABC.

    [See Ex. 9, p. 73.]

PROPOSITION 48. THEOREM.

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by these two sides shall be a right angle.



Let ABC be a triangle; and let the square described on BC be equal to the sum of the squares described on BA, AC:
then shall the angle BAC be a right angle.

Construction. From A draw AD at right angles to AC; I. 11.
and make AD equal to AB. I. 3.
Join DC.

Proof. Then, because AD is equal to AB, *Constr.*
therefore the square on AD is equal to the square on AB.
To each of these add the square on CA;
then the sum of the squares on CA, AD is equal to the sum of the squares on CA, AB.

But, because the angle DAC is a right angle, *Constr.*
therefore the square on DC is equal to the sum of the squares on CA, AD. I. 47.
And, by hypothesis, the square on BC is equal to the sum of the squares on CA, AB;
therefore the square on DC is equal to the square on BC:
therefore also the side DC is equal to the side BC.

Then in the triangles DAC, BAC,

Because { DA is equal to BA, *Constr.*
and AC is common to both;
also the third side DC is equal to the third side BC; *Proved.* }

therefore the angle DAC is equal to the angle BAC. I. 8.
But DAC is a right angle; *Constr.*
therefore also BAC is a right angle. Q. E. D.

EXERCISES ON BOOK I.

ON THE IDENTICAL EQUALITY OF TRIANGLES.

1. If in a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles.

2. If the bisector of the vertical angle of a triangle is also perpendicular to the base, the triangle is isosceles.

3. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

[Produce the bisector, and complete the construction after the manner of I. 16.]

4. If in a triangle a pair of straight lines drawn from the extremities of the base, making equal angles with the sides, are equal, the triangle is isosceles.

5. If in a triangle the perpendiculars drawn from the extremities of the base to the opposite sides are equal, the triangle is isosceles.

6. Two triangles ABC, ABD on the same base AB, and on opposite sides of it, are such that AC is equal to AD, and BC is equal to BD: shew that the line joining the points C and D is perpendicular to AB.

7. ABC is a triangle in which the vertical angle BAC is bisected by the straight line AX: from B draw BD perpendicular to AX, and produce it to meet AC, or AC produced, in E; then shew that BD is equal to DE.

8. In a quadrilateral ABCD, AB is equal to AD, and BC is equal to DC: shew that the diagonal AC bisects each of the angles which it joins.

9. In a quadrilateral ABCD the opposite sides AD, BC are equal, and also the diagonals AC, BD are equal: if AC and BD intersect at K, shew that each of the triangles AKB, DKC is isosceles.

10. If one angle of a triangle be equal to the sum of the other two, the greatest side is double of the distance of its middle point from the opposite angle.

ON PARALLELS AND PARALLELOGRAMS.

11. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected; shew that the bisectors meet at right angles. [I. 29, I. 32.]

12. The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal; and the resulting figure is a rhombus.

13. The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the parallels.

14. A straight line drawn between two parallels and terminated by them, is bisected; shew that any other straight line passing through the middle point and terminated by the parallels, is also bisected at that point.

15. If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal.

16. AB and CD are two given straight lines, and X is a given point in AB: find a point Y in AB such that YX may be equal to the perpendicular distance of Y from CD.

17. ABC is an isosceles triangle; required to draw a straight line DE parallel to the base BC, and meeting the equal sides in D and E, so that BD, DE, EC may be all equal.

18. The straight line drawn through the middle point of a side of a triangle parallel to the base, bisects the remaining side.

19. The straight line which joins the middle points of two sides of a triangle, is parallel to the third side.

20. The straight line which joins the middle points of two sides of a triangle is equal to half the third side.

21. Shew that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which are identically equal.

22. Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.

23. AB, AC are two given straight lines, and P is a given point between them; required to draw through P a straight line terminated by AB, AC, and bisected by P.

24. ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC: shew that BX and DY trisect the diagonal AC.

25. If the middle points of adjacent sides of any quadrilateral be joined, the figure thus formed is a parallelogram.

26. Shew that the straight lines which join the middle points of opposite sides of a quadrilateral, bisect one another.

ON AREAS.

27. Shew that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals. [I. 29, 26.]

28. Bisect a parallelogram by a straight line drawn through a given point.

29. Bisect a parallelogram by a straight line drawn perpendicular to one of its sides.

30. Bisect a parallelogram by a straight line drawn parallel to a given straight line.

31. ABCD is a trapezium in which the side AB is parallel to DC. Shew that its area is equal to the area of a parallelogram formed by drawing through X, the middle point of BC, a straight line parallel to AD. [I. 29, 26.]

32. If two straight lines AB, CD intersect at X, and if the straight lines AC and BC, which join their extremities are parallel, shew that the triangle AXD is equal to the triangle BXC.

33. If two straight lines AB, CD intersect at X, so that the triangle AXD is equal to the triangle XCD, then AC and BD are parallel.

34. ABCD is a parallelogram, and X any point in the diagonal AC produced; shew that the triangles XBC, XDC are equal. [See Ex. 13, p. 64.]

35. If the middle points of the sides of a quadrilateral be joined in order, the *parallelogram* so formed [see Ex. 25] is equal to half the given figure.

MISCELLANEOUS EXAMPLES.

36. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA; if DC is drawn, shew that BCD is a right angle.

37. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

38. From the extremities of the base of a triangle perpendiculars are drawn to the opposite sides (produced if necessary); shew that the straight lines which join the middle point of the base to the feet of the perpendiculars are equal.

39. In a triangle ABC, AD is drawn perpendicular to BC; and X, Y, Z are the middle points of the sides BC, CA, AB respectively: shew that each of the angles ZXY, ZDY is equal to the angle BAC.

40. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the two triangles thus formed are equiangular to one another.

41. If from the middle points of the sides of a triangle perpendiculars be drawn to the sides, shew that they will meet in one point.

42. Shew that the bisectors of the angles of a triangle meet in one point.

43. Shew that the bisectors of two exterior angles of a triangle meet on the bisector of the third angle.

44. Prove that the medians of a triangle meet in one point.

45. In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A, and terminated by the base BC, is less than AB.

46. ABC is a triangle, and the vertical angle BAC is bisected by a straight line which meets the base BC in X; shew that BA is greater than BX, and CA greater than CX. Hence obtain a proof of I. 20.

47. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

48. The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

49. The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.

50. The perimeter of a quadrilateral is greater than the sum of its diagonals.

51. In the figure of I. 47, shew that

(i) the sum of the squares on AB and AE is equal to the sum of the squares on AC and AD.

(ii) the square on EK is equal to the square on AB with four times the square on AC.

(iii) the sum of the squares on EK and FD is equal to five times the square on BC.

52. Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are identically equal.

53. Use the properties of the equilateral triangle to trisect a given finite straight line.

54. Construct a triangle having given the base, one of the angles at the base, and the sum of the remaining sides.

55. Construct a triangle having given the base, one of the angles at the base, and the difference of the remaining sides.

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